Algorithms and Structure:
Set Partitions Under Local Constraints

PhD Thesis

by

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Algoritmus és struktúra: Halmazpartíciók lokális feltételekkel

Doktori (PhD) értekezés

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Kivonat

Algoritmus és struktúra: Halmazpartíciók lokális feltételekkel

A gráfszínezés a gráfelmélet egyik alapvető fontosságú ága, amelyhez számtalan modern műszaki és tudományos alkalmazás kapcsolódik. Ezen gyakorlati problémák tették szükségessé egy sor új típusú színezési feltétel megjelenését is.

A jelen disszertáció első része a Voloshin által 1993-ban bevezetett ,,vegyes hipergráfok"-ra vonatkozó új eredményeket tárgyalja. A szerző bebizonyítja azt az 1995 óta fennálló sejtést, amely a $\mathcal{C}$-perfekt $\mathcal{C}$-hiperfák pontos jellemzését adja; továbbá több év óta nyitott kérdéseket old meg az uniform vegyes hipergráfokra vonatkozóan. A dolgozatban bizonyítást nyer még néhány meglepő komplexitási eredmény is.

A második részben egy új hipergráf-színezési modell kerül bevezetésre ,,stab-hipergráfok" elnevezéssel. A színezési feltételek négy színkorlát-függvény által adhatók meg, amelyek élenként alsó és felső korlátot írnak elő az ott előforduló színek számára és a legnagyobb egyszínű részhalmaz méretére vonatkozóan. Ebben a modellben speciális esetként benne foglaltatik a klasszikus gráf- és hipergráf-színezés, valamint a vegyes hipergráfok színezése is. Továbbá, ahogyan ezt a tézis eredményei több szempontból is alátámasztják, az előzőeknél erősebb modellt kapunk, amely egységes keretet nyújt a partíciós feltételek nem-klasszikus változatainak leírásához is. Stab-hipergráfok segítségével természetes és átlátható modell adható az informatikához és más területekhez tartozó problémák széles körére, amint ez a ,,frekvencia-kiosztási probléma" különböző változataira részletesen is kifejtésre kerül.

A négy színkorlát-függvény közül csak néhányat tekintve, újabb struktúraosztályok nyerhetők. Ezek részletes összehasonlítását és hierarchikus viszonyát is tartalmazza a disszertáció, nagy hangsúlyt helyezve a lehetséges kromatikus polinomok halmazának és bizonyos problémák komplexitási helyzetének változásaira. Szintén bizonyítást nyer a hiperfák központi szerepe a kromatikus polinomokra és a partíciós osztályok lehetséges számára vonatkozóan.

Abstract

Algorithms and structure: Set partitions under local constraints

Graph coloring is a highly developed subject with many modern applications from several fields of science and engineering. These applications have required also the introduction of various non-classical coloring constraints.

The first part of this Thesis contains contributions to the theory of mixed hypergraphs introduced by Voloshin in 1993. The author proves a ten-year-old conjecture concerning the characterization of $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees, and solves long-standing open problems in connection with uniform mixed hypergraphs. Unexpected complexity results are presented, too.

In the second part the new hypergraph coloring model of 'stably bounded' hypergraphs is introduced. The local constraints are expressed by four color-bound functions prescribing lower and upper bounds for the cardinality of largest polychromatic and monochromatic subsets of each hyperedge. This model includes, as particular cases, the vertex colorings of graphs and hypergraphs in the classical sense and the class of mixed hypergraphs. Moreover, as it is pointed out, a much stronger model is obtained, which provides a common frame expressing also non-classical variations of partition constraints. It can be applied for modeling problems of science and engineering in a natural way, as it is demonstrated on several versions of the 'frequency assignment problem'.

The subsets of the introduced four color-bound functions, yield a hierarchy of structure classes. A detailed comparison of these classes is carried out, concentrating mainly on the chromatic polynomials and on the complexity status of certain problems. It is also pointed out that hypertrees play central role regarding chromatic polynomials and the possible number of partition classes.

Zusammenfassung

Algorithmen und Strukturen: Partition von Mengen unter lokalen Beschränkungen

Färbung ist ein hoch entwickeltes Feld der Graphentheorie mit verschiedenen Anwendungen in zahlreichen technischen und wissenschaftlichen Bereichen. Diese Anwendungen brachten auch eine Reihe von neuartigen Beschränkungen für das Färbungsproblem mit sich.

In dem ersten Teil dieser Dissertation werden neue Resultate zu den von Voloshin in 1993 eingeführten gemischten Hypergraphen vorgestellt. Der Autor beweist die seit 1995 bestehende Vermutung über die Charakterisierung von $\mathcal{C}$-perfekten $\mathcal{C}$-Hyperbäumen und gibt Lösungen für einige seit Jahren bestehende Probleme der uniformen gemischten Hypergraphen. In der Abhandlung werden auch einige überraschende Komplexitäts-Resultate vorgestellt.

Im zweiten Teil wird ein neues Modell, „STAB-Hypergraphen" genannt, für Färbung von Hypergraphen eingeführt. Die lokalen Beschränkungen können durch vier Farbenbeschränkungs-Funktionen beschrieben werden, die für jede Hyperkante die untere und obere Grenze der Mächtigkeit der größten einfarbigen und mehrfarbigen Teilmengen angeben. Das vorgestellte Modell beinhaltet die klassische Eckenfärbung von Graphen und Hypergraphen sowie von gemischten Hypergraphen als Spezialfälle. Darüberhinaus ergibt sich ein viel stärkeres Modell, das auch die Betrachtung von nicht-klassischen Varianten der Partitionsbeschränkungen im einheitlichen Rahmen ermöglicht. Die Anwendung der „STAB-Hypergraphen" bietet ein naheliegendes und transparentes Modell für die Analyse einer Vielfalt von Problemen in der Informatik und in anderen wissentschaftlichen Bereichen, was anhand verschiedener Versionen des „Frequenz-Zuordnungsproblems" dargestellt wird.

Die Teilmengen der vorgestellten vier Farbenbeschränkungs-Funktionen ergeben weitere Strukturklassen. Weiterhin wird ein detallierter Vergleich dieser Klassen in der Dissertation durchgeführt, wobei die chromatischen Polynome und die Komplexität bestimmter Probleme im Mittelpunkt stehen. Ferner wird gezeigt, dass Hyperbäume eine zentrale Rolle bei der Bestimmung der chromatischen Polynome und der möglichen Anzahl der Partitionsklassen spielen.

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1 Introduction

In this Thesis we introduce a new model for partitions of set systems, prove many of its fundamental properties, solve some older problems on a more restricted model, and indicate how those structure classes provide a unified description for various problems in mobile communication and other questions in computer science.

The fast development in informatics raises many new types of problems in computer science and leads to new directions in the study of discrete mathematical models. The most obvious natural example is the structure of communication networks, for which the theory of graphs provides an adequate framework. It is beyond the scope of this Thesis to give an extensive account on the various applications of graph theory; but some of them will be mentioned in the sequel, and some will be described in detail in Chapter 9. It should be noted already at this early point, however, that hypergraphs (set systems) provide a more general model for treating many problems in a unified manner.

One of the many ways leading to the concept of hypergraph is to generalize that of graph. Whilst in a graph every edge has two endpoints, in a hypergraph a vertex subset of any cardinality can be viewed as a hyperedge. Formally, hypergraph means a set system on an underlying set called vertex set, and this definition alone does not assume the knowledge of graphs. But viewing a set system as a hypergraph has the advantage that various concepts of graph theory can be adopted for a wider class of structures.

Although the roots of hypergraph theory date back at least to the middle of the nineteenth century (Rev. Kirkman's work on triple systems, 1847), only Berge's research monograph [15] was the first one that devoted a separate part to discussing hypergraphs. It had been clear, however, that hypergraphs are not merely generalizations of graphs but they really capture a higher level of abstraction and lead to solutions of problems that cannot be attacked by graph-theoretic methods alone.

Starting from the end of the nineteenth century, vertex coloring has been one of the most widely studied and applied areas of graph theory. In the classical sense, a proper vertex coloring of a graph is a function assigning colors to the vertices in such a way that any two adjacent vertices have different colors (traditionally denoted with natural numbers). An equivalent definition from a different viewpoint is to partition the vertex set into subsets in which each pair of vertices is nonadjacent. Concerning what is known and what is not known on the theoretical side, the reader can find a wealth of information in Jensen and Toft's book [28]; and many applications are mentioned in the papers by Roberts [48, 49].

The classical vertex coloring of a hypergraph was introduced in 1966 by Erdős and Hajnal [22]. Analogously to graph coloring, a vertex coloring of a hypergraph is considered proper if each hyperedge contains at least two vertices with distinct colors. Equivalently, one looks for a vertex partition into subsets that contain no hyperedges. In this way, a global partition is required to satisfy local constraints described in terms of hyperedges.

In the middle of the 1990's, Voloshin extended the concept by introducing the idea of mixed hypergraphs [57, 58]. This novel type of hypergraph coloring — that imposes a further kind of local constraints — turned out to be a fruitful generalization of classical coloring. In the last decade the theory grew rapidly, more than 150 papers related to this area have been published. Due to page limitation, here we cannot summarize much of the theory; only the most important issues in connection with our new results will be mentioned in the text. A large amount of results can be found in Voloshin's research monograph [59], in the recent survey by Tuza and Voloshin [53], and on the regularly updated web site [60].

The first part of the Thesis is based on the papers [2, 3, 7, 8] and contains contributions to the theory of Voloshin's mixed hypergraphs. We present polynomial-time algorithms, settle the time complexity of several algorithmic problems, give the answer to a fundamental extremal question, and prove characterization theorems. Among the latter, the most notable one is the necessary and sufficient condition for a $\mathcal{C}$-hypertree to be $\mathcal{C}$-perfect, that was conjectured by Voloshin in 1995 and is proved here.

In the second part of the Thesis we introduce new models of hypergraph coloring, based on a series of papers [1, 4, 5, 6]. Those color-bounded and stably bounded hypergraphs admit more flexibility in the local constraints that can be imposed on the vertex partitions allowed, and so they generalize previous concepts: mixed hypergraphs and a coloring problem studied recently by Drgas-Burchardt and Łazuka [19]. We discuss the similarities and differences between our general models and the more particular earlier ones. Important issues are chromatic polynomials, the role of hypertrees, and problems having different algorithmic complexities in various kinds of models. Beside the study of structural properties, we put much emphasis on designing polynomial-time algorithms when the problem in question admits an efficient solution.

Before surveying the theory of mixed, color-bounded and stably bounded hypergraphs and describing our new results, unfortunately it is unavoidable to begin with definitions that will be used throughout this work.

1.1 Basic definitions

- A *mixed hypergraph* is a triple, $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the vertex set, and $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of X. It is assumed that X is finite[1] and $|H| \geq 2$ holds for all $H \in \mathcal{C} \cup \mathcal{D}$.

[1]As usual, the cardinality of X will be denoted by n throughout this work.

- The members of $\mathcal{C}$ and $\mathcal{D}$ are called *$\mathcal{C}$-edges* and *$\mathcal{D}$-edges*, respectively. A set $H \in \mathcal{C} \cap \mathcal{D}$ is called a *bi-edge*.

- A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a *$\mathcal{C}$-hypergraph* if $\mathcal{D} = \emptyset$, whilst in the case when $\mathcal{C} = \emptyset$ we obtain a *$\mathcal{D}$-hypergraph*[2]. A *bi-hypergraph* is a mixed hypergraph with $\mathcal{C} = \mathcal{D}$.

The distinction between $\mathcal{C}$-edges and $\mathcal{D}$-edges becomes substantial when colorings are considered.

- A *proper vertex coloring* — or for short, a *coloring* — of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a mapping φ from the vertex set X into a set of colors, where each $\mathcal{C}$-edge has at least two vertices with a *common* color and each $\mathcal{D}$-edge has at least two vertices with *distinct* colors. Without loss of generality, the colors will be denoted by the positive integers $1, 2, \ldots, k$.

- Each coloring φ of a mixed hypergraph $\mathcal{H}$ induces a *color partition* $X_1 \cup \cdots \cup X_k = X$, where the partition classes are the inclusion-wise maximal monochromatic subsets of X under φ.

- We shall use the term *k-coloring* for a coloring with precisely k nonempty color classes. Note that in [59] these are called *strict* k-colorings, but in the present context we do not need such a distinction.

- For $k = 1, 2, \ldots, n$ the number of color partitions of X with precisely k nonempty classes will be denoted by r_k. (Here we disregard the renumberings of colors.) The n-tuple $(r_1, r_2, \ldots, r_n)$ is termed the *chromatic spectrum* of $\mathcal{H}$. We consider two chromatic spectra $(p_1, p_2, \ldots, p_j)$ and $(r_1, r_2, \ldots, r_k)$ to be equal if one is a prefix of the other, and all the remaining entries of the other sequence are zeros. That is, assuming $j \leq k$, we require $p_i = r_i$ for all $1 \leq i \leq j$ and $r_i = 0$ for all $j < i \leq k$. Adopting this point of view, we usually write chromatic spectrum in the form omitting all zeros from the end. If two hypergraphs have equal chromatic spectra, they are said to be *chromatically equivalent*.

- The *cromatic polynomial* $P(\mathcal{H}, \lambda)$ of a mixed hypergraph $\mathcal{H}$ is, by definition, the polynomial whose value at each positive integer k is equal to the number of proper colorings of $\mathcal{H}$ with *at most k distinguished* colors; that is, the number of mappings $\varphi : X \to \{1, \ldots, k\}$ whose color partition $(\varphi^{-1}(1), \ldots, \varphi^{-1}(k))$ induced on X is proper for $\mathcal{H}$. (Here some of the sets $\varphi^{-1}(i)$ are allowed to be empty.)

Let us emphasize some substantial differences between r_k and the value of $P(\mathcal{H}, \lambda)$ at $\lambda = k$. The former does not count permutations of colors to be distinct, while the

[2]Vertex coloring of $\mathcal{D}$-hypergraphs will correspond to hypergraph coloring in the classical sense.

latter does; moreover, the former only takes into consideration the colorings with *precisely* k colors.

- The *feasible set* of $\mathcal{H}$, denoted by $\Phi(\mathcal{H})$, is the set of possible numbers of colors in a coloring: $\Phi(\mathcal{H}) = \{k \mid r_k \neq 0\}$.

In classical hypergraph coloring every hypergraph has a proper coloring with $n = |X|$ colors. Similarly, every $\mathcal{C}$-hypergraph can be properly colored using only one color. But there exist mixed hypergraphs which have no proper colorings at all.[3]

- A mixed hypergraph $\mathcal{H}$ is *colorable* if $\Phi(\mathcal{H}) \neq \emptyset$; and otherwise it is called *uncolorable.*

- A mixed hypergraph $\mathcal{H}$ is *uniquely colorable* — UC-graph or UC, for short — if it has precisely one proper coloring disregarding the renumberings of colors.

- A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is *UC-orderable* if there exists a vertex-order $x_1, x_2, \ldots, x_n$ on the vertex set X with the following property : for each $1 \leq i \leq n$, the subhypergraph $\mathcal{H}_i$ induced by $\{x_1, \ldots, x_i\}$ is uniquely colorable. Such a vertex-order on X will be termed a *UC-order*.

- A UC-graph is called *uniquely UC-orderable* — UUC-graph, or UUC, for short — if it has just one UC-order apart from the transposition of the first two vertices.[4]

- Assuming that $\mathcal{H}$ is colorable, the largest and smallest possible numbers of colors in a proper coloring are termed the *upper chromatic number* and *lower chromatic number* of $\mathcal{H}$, respectively. In notation, $\overline{\chi}(\mathcal{H}) = \max \Phi(\mathcal{H})$ and $\chi(\mathcal{H}) = \min \Phi(\mathcal{H})$. If $\mathcal{H}$ is uncolorable, for technical reasons it is convenient to define these values to be $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = 0$.

It is quite natural to ask whether a colorable hypergraph has k-colorings for all integers k between its lower and upper chromatic number. The answer is trivially positive for classical ($\mathcal{D}$-) and $\mathcal{C}$-hypergraphs, but it does not hold for mixed hypergraphs in general, as it was proved by Jiang et al. in [29]. This fact made it necessary to introduce the following term.

- A *gap* in the chromatic spectrum of $\mathcal{H}$ (or a gap in $\Phi(\mathcal{H})$), is an integer $k \notin \Phi(\mathcal{H})$ such that $\min \Phi(\mathcal{H}) < k < \max \Phi(\mathcal{H})$. If $\Phi(\mathcal{H})$ has no gaps, then the spectrum or feasible set is termed *continuous* or *gap-free.* More generally, a *gap of size* ℓ in $\Phi(\mathcal{H})$ means ℓ consecutive integers that are all missing from $\Phi(\mathcal{H})$, larger than $\chi(\mathcal{H})$ and smaller than $\overline{\chi}(\mathcal{H})$.

[3] The simplest example is the mixed hypergraph which consists of a 2-element $\mathcal{C}$-edge and a 2-element $\mathcal{D}$-edge on the same vertex pair.

[4] The smallest UC-orderable non-UUC-graph consists of three vertices mutually joined by 2-element $\mathcal{D}$-edges, that is the simple graph K_3.

Let us continue with some structural conditions.

- For a hypergraph $\mathcal{H}$, a *host graph* is a graph G on the same vertex set as $\mathcal{H}$, and such that every hyperedge induces a *connected* subgraph in G. Depending on the type of G, particular terminology is used for $\mathcal{H}$:
  - If G is a path, then $\mathcal{H}$ is called an *interval hypergraph*.
  - If G is a tree, then $\mathcal{H}$ is called a *hypertree*.
  - If G is a cycle, then $\mathcal{H}$ is called a *circular hypergraph*.
- A hypergraph is called *r-uniform* if all of its hyperedges have exactly r vertices, and *d-regular* if each vertex is incident with precisely d hyperedges.
- The *dual* of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ is obtained when we represent each edge $E_i \in \mathcal{E}$ by a new vertex x_i^* and each vertex $x_j \in X$ by a new edge E_j^*, while keeping the structure of incidences unchanged; i.e., $x_i^* \in E_j^*$ if and only if $x_j \in E_i$.
- A hypergraph is *linear* if any two of its edges have at most one vertex in common.
- A *subhypergraph* of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ means a vertex set $Y \subseteq X$ together with *some* hyperedges H of $\mathcal{H}$ for which $H \subseteq Y$ holds.
- An *induced subhypergraph* of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ means a vertex set $Y \subseteq X$ together with *all* hyperedges H of $\mathcal{H}$ such that $H \subseteq Y$.
- The *$\mathcal{C}$-stability number* $\alpha_C(\mathcal{H})$ is the largest cardinality of a vertex subset in $\mathcal{H}$ not containing any $\mathcal{C}$-edges.

If we consider a $\overline{\chi}$-coloring of $\mathcal{H}$ and choose exactly one vertex from each color class, we obtain a $\overline{\chi}(\mathcal{H})$-element vertex subset not containing any $\mathcal{C}$-edges. Consequently, the inequality $\alpha_C(\mathcal{H}) \geq \overline{\chi}(\mathcal{H})$ holds for every mixed hypergraph.

- A mixed hypergraph $\mathcal{H}$ is *$\mathcal{C}$-perfect* if the condition $\alpha_C(\mathcal{H}') = \overline{\chi}(\mathcal{H}')$ is satisfied by each induced subhypergraph $\mathcal{H}'$ of $\mathcal{H}$.
- A *monostar* is a mixed hypergraph in which the intersection of $\mathcal{C}$-edges is precisely one vertex.
- A *polystar* is a mixed hypergraph with at least two $\mathcal{C}$-edges, in which the intersection Y of the $\mathcal{C}$-edges is nonempty, and every vertex pair in Y forms a $\mathcal{D}$-edge. (The particular case of $|Y| = 1$ means a monostar.)
- A *bistar* is a mixed hypergraph in which the intersection of $\mathcal{C}$-edges contains a pair of vertices, say x, y, such that $\{x, y\}$ is not a $\mathcal{D}$-edge.

- A *cycloid*, denoted by $\mathcal{C}_n^r$, is an r-uniform $\mathcal{C}$-hypergraph with $n > r$ vertices $x_1, \ldots, x_n$ and n $\mathcal{C}$-edges of the form $\{x_i, x_{i+1}, \ldots, x_{i+r-1}\}$, where subscript addition is taken modulo n and $i = 1, \ldots, n$.

1.2 Results on mixed hypergraphs

Next, we describe our new results on mixed hypergraphs, and their background in the literature. In the title of each part below, we indicate the paper in which the corresponding theorems are published.

Feasible sets of r-uniform mixed hypergraphs [3]. It is clear that a finite set S of positive integers is a feasible set of some $\mathcal{C}$-hypergraph if and only if S contains the number 1 and it is gap-free; i.e., it is of the form $\{1, 2, \ldots, k\}$ for some natural number k. In [29] the possible feasible sets of mixed hypergraphs were completely characterized. The result is quite surprising: for every set S of positive integers not containing 1, there exists a mixed hypergraph whose feasible set is S.

Later it was also proved by Král' [32] that arbitrarily prescribing a sequence of nonnegative integers $r_2, r_3, \ldots, r_k$, there exists a non-1-colorable mixed hypergraph whose chromatic spectrum is $(r_1 = 0, r_2, r_3, \ldots, r_k)$.

For restricted classes of mixed hypergraphs, too, the possible feasible sets were investigated and characterized. Particularly, it was shown for interval mixed hypergraphs (Jiang et al. [29]), for mixed hypertrees (Král' et al. [34]), for circular mixed hypergraphs (Král', Kratochvíl and Voss [35]) and for mixed hypergraphs with maximum vertex degree two (Král', Kratochvíl and Voss [36]) that there is no gap in their feasible sets. However, for bi-hypergraphs in general and for r-uniform mixed hypergraphs the characterization of possible feasible sets was an open problem.

In Chapter 2, we solve both problems for all values of r in a constructive way.

It is easy to see that the following two conditions must hold if the r-uniform hypergraph has at least one hyperedge:

- If the r-uniform mixed hypergraph is 1-colorable (i.e. $\mathcal{C}$-hypergraph), then the feasible set is gap-free.

- If the r-uniform hypergraph has an ℓ-coloring for some $\ell < r - 1$, then it is also colorable with precisely k colors for every $\ell < k \leq r - 1$.

Our Theorem 1 states that these two conditions are sufficient, too. In Chapter 2 an appropriate r-uniform mixed hypergraph is constructed for each feasible set S satisfying the two conditions above. Moreover, the possible feasible sets of r-uniform bi-hypergraphs (containing at least one hyperedge) are completely characterized by the second condition and by the trivial restriction $1 \notin S$ (Chapter 2). As a consequence, we obtain that S is a feasible set of some bi-hypergraph with at least one hyperedge if and only if $1 \notin S$.

Uniform $\mathcal{C}$-hypergraphs admitting only 'under-size' colorings [8]. Important extremal problems arise when the minimum number of hyperedges is considered, for which there exists some mixed hypergraph having n vertices and a prescribed feasible set. These questions seem to be quite hard and in many cases they are connected to Turán-type problems.[5]

Voloshin asked (Problem 11 of [58] and Problem 2 on page 43 of [59]) for the minimum number of hyperedges in an r-uniform $\mathcal{C}$-hypergraph $\mathcal{H}$ of order n for which $\overline{\chi}(\mathcal{H}) < k$.

Considering an r-uniform $\mathcal{C}$-hypergraph on an n-element vertex set ($n \geq r$), every coloring with at most $r-1$ colors (we can say, every 'under-size' coloring) is trivially proper, since multicolored edges cannot arise. The minimum number of r-element $\mathcal{C}$-edges in a $\mathcal{C}$-hypergraph of order n, not admitting colorings in addition to trivial ones, is denoted by $f(n,r)$. For this minimum number, a lower bound was known. This gives the exact value if $r = 3$ as it was shown recently by a recursive construction. For the 4-uniform case the previously known best upper bound was asymptotically $\frac{n^3}{8}$, whilst the lower bound is only $\frac{n^3}{12}$. For greater values of r there were not known any non-trivial upper bounds.

We prove in Chapter 3 that for each $r > 3$, there exist infinitely many values of n for which the lower bound is not tight. This makes it interesting to find asymptotically tight upper bounds. We note further that $f(n,n-2) = \binom{n-2}{2} - \mathrm{ex}(n,\{C_3,C_4\})$ holds, where the last term is the Turán number for graphs of girth five.[6] This fact indicates that the exact determination of $f(n,r)$ is far beyond reach to our present knowledge.

Our new result, described in Chapter 3, is an upper bound for this extremal value $f(n,r)$. Answering a ten-year-old problem of Voloshin, it yields asymptotically tight solution for each fixed r and, beyond that, for all $r = o(\sqrt[3]{n})$ as $n \to \infty$.

Although we do not discuss it in detail in this Thesis, let us mention that we study this extremal problem in the following more general setting in another manuscript [9]. We say that a subset (hyperedge) H *crosses* a k-partition $X_1 \cup X_2 \cup \cdots \cup X_k$ of X if it intersects precisely $\min(|H|,k)$ of the k partition classes. We consider the smallest possible number of r-element subsets of an n-element set X, such that each k-partition of X is crossed by at least one of the selected subsets, and denote this minimum number by $f(n,k,r)$. If $k = r$, the value $f(n,r)$ defined above is obtained. If $k > r$, the value $f(n,k,r)$ is equal to the minimum number of $\mathcal{C}$-edges in an r-uniform $\mathcal{C}$-hypergraph $\mathcal{H}$ of order n, for which $\overline{\chi}(\mathcal{H}) < k$. In the third case, when $k < r$, the value $f(n,k,r)$ can be interpreted analogously in terms of color-bounded hypergraphs.

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[5] In a Turán-type extremal problem, 'forbidden' graphs $G_1, G_2, \ldots, G_k$ are given, and we want to determine the maximum number of edges in a graph of order n, which contains no subgraph isomorphic to any of $G_1, G_2, \ldots, G_k$. This maximum value is denoted by $\mathrm{ex}(n,\{G_1, G_2 \ldots, G_k\})$. The definition can be extended to uniform hypergraphs, too, in a natural way.

[6] That is, the maximum number of edges in a graph on n vertices that contains no subgraph isomorphic to the 3-cycle and the 4-cycle.

Interestingly enough, the investigation of this extremal function leads us to intensively studied areas of combinatorics, such as Balanced Incomplete Block Designs[7] and Turán-type extremal problems on graphs and hypergraphs.

$\mathcal{C}$-perfectness [7]. A beautiful part of graph coloring theory deals with the class of perfect graphs. It is immediate by definition that the chromatic number $\chi(G)$ of any graph G is at least as large as the maximum clique size $\omega(G)$. The graph G is called perfect if the equality $\chi(G') = \omega(G')$ holds for each induced subgraph G' of G. The class of perfect graphs contains many interesting and important subclasses, and on the other hand it admits polynomial-time optimization algorithms for some central problems that are NP-hard in general.

The analogous notion of $\mathcal{C}$-perfectness for mixed hypergraphs was introduced by Voloshin in [58]. Similarly to graphs, it is expected that many hard algorithmic problems become efficiently solvable when the input is restricted to the class of $\mathcal{C}$-perfect mixed hypergraphs.

Since $\mathcal{C}$-perfectness is a hereditary property (i.e., if a hypergraph is $\mathcal{C}$-perfect, then so is each of its induced subhypergraphs), the main goal is to determine the inclusion-wise minimal $\mathcal{C}$-imperfect hypergraphs.

It was stated by Voloshin in [58] that in the class of $\mathcal{C}$-hypertrees all the minimal $\mathcal{C}$-imperfect hypergraphs are monostars. Taking into account that every monostar is imperfect, this is equivalent to the following characterization: a $\mathcal{C}$-hypertree is $\mathcal{C}$-perfect if and only if it contains no monostar as an induced subhypergraph. But later it was discovered (cf. [59, Section 5]) that the original proof of sufficiency does not work.

This ten-year-old problem is solved here in Chapter 4. Our main result concerning $\mathcal{C}$-hypertrees is an algorithmic proof, which implies that the characterization of $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees, as proposed in [58], is valid indeed. Furthermore, this characterization is extended under certain conditions for mixed hypertrees, too.

From the other side, there was a strong expectation that $\mathcal{C}$-perfect mixed hypertrees can be recognized and $\overline{\chi}$-colored in polynomial time. The former expectation is refuted in the Thesis, showing that the recognition problem of $\mathcal{C}$-perfect ones is co-NP-complete already for $\mathcal{C}$-hypertrees. As regards the latter expectation, we present a polynomial-time $\overline{\chi}$-coloring algorithm, which can be applied for $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees, and also for a wider subclass of $\mathcal{C}$-perfect mixed hypertrees.

Orderings of uniquely colorable hypergraphs [2]. Every graph is colorable, and those having only one proper color partition — complete graphs — play a central

[7] A Balanced Incomplete Block Design (BIBD), also called a Steiner system of order v and index λ — often denoted by $S_\lambda(t,k,v)$, where $\lambda \geq 1$ and $2 \leq t < k \leq v$ — consists of a set X of v elements and a collection of k-element subsets of X, called blocks, such that each t-element subset of X appears in exactly λ blocks. For $\lambda = 1$, one usually writes $S(t,k,v)$.

role in graph theory. In classical hypergraph coloring there appear no other types of uniquely colorable hypergraphs. But in the class of mixed hypergraphs there exists a wide range of uniquely colorable systems. Their structure is so unrestricted that every colorable mixed hypergraph can be embedded into some uniquely colorable one as an induced subhypergraph (Tuza, Voloshin and Zhou [56]). In accordance with this, the recognition problem of uniquely colorable mixed hypergraphs is intractable, and its time complexity is co-NP-complete, if the input is restricted to colorable mixed hypergraphs with a proper coloring given in the input.

It had been expected for several years, however, that the more restricted *UC-orderable* hypergraphs could be recognized efficiently. Our Theorem 7 disproves this expectation, stating that the recognition problem of UC-orderable hypergraphs remains NP-complete even if it is restricted to uniquely colorable ones and the proper coloring is given in the input.

By another result of Chapter 5, the possible color sequences of uniquely UC-orderable mixed hypergraphs are characterized, and our method also yields a linear-time test for the recognition of their possible color sequences. Moreover, we construct a uniquely UC-orderable mixed hypergraph with minimum number of hyperedges and having several further extremal properties, for each possible color sequence.

1.3 New models: Color-bounded and stably bounded hypergraphs

Color-bounded hypergraphs. In Chapters 6 and 7 we introduce and study a new model of hypergraph coloring, termed color-bounded hypergraphs. The heart of the matter is that each hyperedge E_i is associated with a lower color bound s_i and an upper color bound t_i. A vertex coloring is considered proper if each hyperedge E_i gets at least s_i and at most t_i different colors.

Our model has been inspired, on the one hand, by the recent work of Drgas-Burchardt and Łazuka [19], who considered the case of arbitrarily specified lower bounds s_i but without upper bounds (what is equivalent to writing $t_i = |E_i|$ for all $i \leq m$); and, on the other hand, by the area of mixed hypergraphs. In the latter, the $\mathcal{C}$-edges and $\mathcal{D}$-edges can be characterized as $(s_i, t_i) = (1, |E_i| - 1)$ and $(s_i, t_i) = (2, |E_i|)$, respectively. The bi-edges are then those with $(s_i, t_i) = (2, |E_i| - 1)$; hence, these notions have a natural and unified description in our model. The traditional concept of 'proper vertex coloring' in the usual hypergraph-theoretic sense can be described with $(s_i, t_i) = (2, |E_i|)$ for all edges.

Now, we introduce the concept of color-bounded hypergraph more formally.

- A *color-bounded hypergraph* is a four-tuple $\mathcal{H} = \{X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}\}$ where $(X, \mathcal{E})$ is a hypergraph (set system) with vertex set X and edge set $\mathcal{E}$, and $\boldsymbol{s} : \mathcal{E} \to \mathbb{N}$ and $\boldsymbol{t} : \mathcal{E} \to \mathbb{N}$ are integer-valued functions. We assume throughout that

$$X = \{x_1, \ldots, x_n\}, \quad \mathcal{E} = \{E_1, \ldots, E_m\}$$

and that

$$1 \le \boldsymbol{s}(E_i) \le \boldsymbol{t}(E_i) \le |E_i| \qquad \text{for all} \;\; 1 \le i \le m.$$

To simplify notation, we write

$$s_i := \boldsymbol{s}(E_i), \quad t_i := \boldsymbol{t}(E_i), \quad s := \max_{E_i \in \mathcal{E}} s_i.$$

- A (proper) *vertex coloring* of a color-bounded hypergraph $\mathcal{H} = \{X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}\}$ is a mapping $\varphi : X \to \mathbb{N}$ such that the number of colors occurring in E_i satisfies

$$s_i \le |\varphi(E_i)| \le t_i \qquad \text{for all} \;\; 1 \le i \le m.$$

- The concepts of *color-partition, k-coloring, chromatic spectrum, chromatic polynomial, feasible set, unique coloring, lower* and *upper chromatic number, gap, induced* and *non-induced subhypergraph* have already been defined for mixed hypergraphs, and we shall use them analogously for color-bounded hypergraphs without rewriting the definitions.

Results on color-bounded hypergraphs [4, 5]. It turns out that color-bounded hypergraphs provide not just a common generalization of the earlier coloring concepts, but in fact a much stronger model is obtained. This is demonstrated in the results of Section 6.3 on the possible numbers of colors in a proper coloring if the cardinality of X is fixed, and of Section 6.4 on unique $(n-1)$-colorability; and partly of Section 6.5, too, concerning 2-regular hypergraphs.

Significant differences between color-bounded and mixed hypertrees are explored further in Chapter 7. For a colorable mixed hypertree $\mathcal{T}$, the lower chromatic number is at most two and the feasible set is always gap-free. We shall prove that in the case of color-bounded hypertrees not only the lower chromatic number, but also the difference $\chi - s$ can be arbitrarily large and there can occur a gap of any size. Furthermore, as it is stated in our Theorem 16, hypertrees represent nearly all color-bounded hypergraphs with respect to possible feasible sets.

Another striking difference appears when the question of colorability is considered. Whilst for mixed hypertrees the decision problem of colorability can be solved in linear time, the analogous problem is intractable already for 3-uniform color-bounded hypertrees (Theorem 18).

On the other hand, we identify some subclasses of hypertrees whose feasible sets contain no gaps (Theorems 15 and 17). In particular, the chromatic spectrum of interval hypergraphs is gap-free (Theorem 14). In Section 7.6 we also prove that the chromatic spectrum of circular hypergraphs is fairly restricted, though this wider class behaves differently from interval hypergraphs with respect to the lower chromatic number.

As regards methodology, an essential tool called *Recoloring Lemma* is presented in Section 7.1. It is then applied in several algorithmic proofs of later sections.

Stably bounded hypergraphs [6]. In Chapter 8 we introduce and study a more general structure class that we call stably bounded hypergraphs. In this model, every hyperedge is associated with four bounds. The bounds s_i and t_i are responsible for the minimum and maximum cardinality of the largest polychromatic subset of the edge, whilst the two other bounds prescribe that the largest number of vertices having the same color inside the edge is at least a_i and at most b_i. The phrase 'stably bounded' hypergraph may be viewed as an alternative rewritten form of '$(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$-ly bounded'.

Next, the main concepts are defined more formally.

- A *stably bounded hypergraph* is a six-tuple $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$, where

  $$\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b} : \mathcal{E} \to \mathbb{N}$$

  are given integer-valued functions on the edge set. To simplify notation, we define

  $$s_i := \boldsymbol{s}(E_i), \qquad t_i := \boldsymbol{t}(E_i), \qquad a_i := \boldsymbol{a}(E_i), \qquad b_i := \boldsymbol{b}(E_i)$$

  and assume throughout that the inequalities

  $$1 \le s_i \le t_i \le |E_i|, \qquad 1 \le a_i \le b_i \le |E_i|$$

  are valid for all edges E_i. We shall refer to $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$ as *color-bound functions*, and to s_i, t_i, a_i, b_i as *color-bounds* on edge E_i.

- Given a coloring function $\varphi : X \to \mathbb{N}$, a set $Y \subseteq X$ is *monochromatic* if $\varphi(y) = \varphi(y')$ for all $y, y' \in Y$; and Y is said to be *polychromatic* (multicolored) if $\varphi(y) \ne \varphi(y')$ for any two distinct $y, y' \in Y$. The largest cardinality of a monochromatic and polychromatic subset of Y will be denoted by $\mu(Y)$ and by $\pi(Y)$, respectively.

- A (proper) *coloring* of $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ is a mapping $\varphi : X \to \mathbb{N}$ such that

  $$s_i \le \pi(E_i) \le t_i \qquad \text{and} \qquad a_i \le \mu(E_i) \le b_i \qquad \text{for all} \qquad E_i \in \mathcal{E}.$$

- The terms *color-partition, k-coloring, chromatic spectrum, chromatic polynomial, feasible set, unique coloring, lower* and *upper chromatic number, gap, induced* and *non-induced subhypergraph* will be used for stably bounded hypergraphs analogously to mixed and color-bounded ones.

In Chapter 8 we give a detailed analysis of the relations among the four color-bound functions. The subsets of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$, as combinations of nontrivial conditions on colorability, form a hierarchy with respect to the strength of models concerning vertex coloring. In a way, the pair $(\boldsymbol{s}, \boldsymbol{a})$ is universal; but, interestingly enough, the partial order among the classes is not always the same, as it may depend on the aspect of comparing the allowed colorings. Our results indicate that concerning the possible numbers of colors on a given number of vertices, the more restrictive function is the *monochromatic* upper bound $\boldsymbol{b}$ (cf. Theorems 20 and 21), while with respect to the number of color partitions in general the stronger restriction is the *polychromatic* upper bound $\boldsymbol{t}$ (see Section 8.4).

Although the decision problem whether a hypergraph admits any proper coloring is NP-complete for all nontrivial combinations of the conditions, nevertheless some algorithmic questions exhibit further substantial differences among the color-bound types. This fact is demonstrated concerning unique colorability in Section 8.5. On the other hand, there are subclasses of stably bounded hypergraphs that admit efficient coloring algorithms.

Chromatic polynomials [4]. The characterization of chromatic polynomials for non-1-colorable hypergraphs is discussed in Section 6.2. This is a new result even for mixed hypergraphs, and it is proved to be valid for color-bounded and stably bounded hypergraphs, too. Furthermore, this characterization can be extended to the more general model of pattern hypergraphs without any restrictions (see definition in Section 6.2); i.e., non-1-colorability is not assumed in that case.

1.4 Applications for problems in informatics

The new structure classes studied in the Thesis provide a general framework for modeling problems from real life. In Chapter 9, concentrating on the field of informatics, we discuss several possible applications in detail.

By definition, in a stably bounded hypergraph we can prescribe the number of colors occurring inside each hyperedge and we can take bounds for the cardinality of the largest monochromatic subset of each edge. But in many applications we have restrictions concerning the number of occurrences of *fixed types*; that is, of fixed colors. It is shown that a stably bounded hypergraph can be supplemented with new vertices (corresponding to the colors) and edges such that the obtained hypergraph expresses the above type of constraints, too.

- The resource allocation problem appears in informatics in several forms. This means a mapping of tasks or processing. The requirements on the (in)compatibility and the number of occurrences can be efficiently expressed using stably bounded hypergraphs.

- Problems in connection with dependability and fault tolerance of IT systems can also be modeled by stably bounded hypergraphs. Here the typical constraints concern the least number of identical resources (replicas) to assure a given level of fault tolerance, whilst an upper bound expresses a cost limit.

- In general sense, the frequency assignment problem means assigning frequencies to the transmitters so that excessive interference is avoided. This problem appears in different forms when concerns mobile telephone networks, radio and television broadcasting, or satellite communication. These different forms yield different constraints for frequency assignment and hence, they inspired different non-classical versions of graph coloring such as distance-labeling, T-coloring, and their more general versions. We will point out that all these varieties can be described in a unified and natural way, using stably bounded hypergraphs.

- Furthermore, we give examples of possible applications regarding the fields of service-oriented architecture, scheduling of file transfers, and data access in parallel memory.

It is worth noting that in the monograph [59] some applications of mixed hypergraphs are discussed from the fields of molecular biology and genetics of populations. Concerning the application of S-hypergraphs there can be found economical examples in the paper [19].

2 Feasible sets of uniform mixed hypergraphs

Concerning graph and hypergraph coloring in the classical sense, we have only one type of constraints. Namely, no edge can have its endpoints with a common color and no hyperedge can have all its vertices with the same color. These requirements can easily be satisfied, since we trivially obtain a proper coloring if every vertex is labeled with a dedicated color. On the other hand, if we have a proper coloring with fewer than $n = |X|$ colors, then any color class containing more than one vertex can be split into two non-empty parts and the coloring conditions remain fulfilled. Thus, for any k between the (lower) chromatic number and n there exists some coloring using precisely k colors. This results in a transparent structure of possible feasible sets. For any positive integers $2 \le a \le b$, we can easily construct graphs or hypergraphs which have k-colorings if and only if $a \le k \le b$. This situation does not change essentially even if it is prescribed that every hyperedge is of the same size r. (More precisely, the only additional condition is $a \le \lfloor \frac{n}{r-1} \rfloor = \lfloor \frac{b}{r-1} \rfloor$.) But the above simple structure of traditional colorings can become disadvantageous when we have to model a problem with a more complex system of constraints.

The notion of mixed hypergraph allows the usage of two opposite conditions: we can require for some fixed groups ($\mathcal{C}$-edges) that each of them should contain two elements labeled identically, while the traditional coloring constraint concerns the $\mathcal{D}$-edges; that is, the latter have to contain two elements labeled differently. By the simultaneous presence of $\mathcal{C}$- and $\mathcal{D}$-edges, a more complex structure with surprising features is obtained. A mixed hypergraph can be uncolorable; or, if it is colorable, the possible numbers of used colors may not form a 'continuous' interval at all (i.e., in their feasible sets there can occur an unrestricted number of 'gaps' with unrestricted sizes). On the one hand, these properties indicate the fact that mixed hypergraphs are applicable for modeling a wide range of practical problems. On the other hand, it is important to study in subclasses of mixed hypergraphs, whether these 'irregular' properties can occur on their members. These results can help in the selection of an appropriate model for a given practical problem. Here we will characterize the possible feasible sets of r-uniform mixed and bi-hypergraphs, answering two open questions of this field.

2.1 Characterization theorem

It is readily seen that if $1 \in \Phi(\mathcal{H})$, then $\mathcal{H}$ cannot have any $\mathcal{D}$-edges, therefore in this case the feasible set $\Phi(\mathcal{H})$ necessarily is gap-free; and vice versa, any gap-free 'interval' $\{1, \dots, k\}$ is a feasible set of some mixed ($\mathcal{C}$-) hypergraph. On the other hand, for the case $1 \notin \Phi(\mathcal{H})$, Jiang *et al.* proved in [29] that for any finite set S of integers greater than 1 there exists a mixed hypergraph $\mathcal{H}$ with $\Phi(\mathcal{H}) = S$.

But the corresponding problem for bi-hypergraphs in general and r-uniform mixed hypergraphs was open for several years.

In this chapter we consider r-uniform mixed hypergraphs, i.e. those with $|C| = |D| = r$ for all $C \in \mathcal{C}$ and all $D \in \mathcal{D}$, with a fixed integer $r \geq 3$. Our main result regarding possible feasible sets is the following characterization:

Theorem 1. *Let $r \geq 3$ be an integer, and S a finite set of natural numbers. There exists a colorable r-uniform mixed hypergraph $\mathcal{H}$ with $\Phi(\mathcal{H}) = S$ and $|\mathcal{C}| + |\mathcal{D}| \geq 1$ if and only if*

(i) $\min(S) \geq r$, *or*

(ii) $2 \leq \min(S) \leq r-1$ *and S contains all integers between* $\min(S)$ *and $r-1$, or*

(iii) $\min(S) = 1$ *and $S = \{1, \ldots, \bar{\chi}\}$ for some natural number $\bar{\chi} \geq r-1$.*

Moreover, S is the feasible set of some r-uniform bi-hypergraph with $\mathcal{C} = \mathcal{D} \neq \emptyset$ if and only if it is of type (i) or (ii).

If $r = 3$, then (i) and (ii) together allow any set not containing the element 1. Hence, a characterization for bi-hypergraphs can be concluded.

Corollary 1. *A finite set S of positive integers is the feasible set of some bi-hypergraph with at least one bi-edge if and only if $1 \notin S$.*

Moreover, we obtain

Corollary 2. *For every mixed hypergraph $\mathcal{H}$ with $\bar{\chi}(\mathcal{H}) > 1$ there exists a 3-uniform mixed hypergraph $\mathcal{H}_3$ such that $\Phi(\mathcal{H}) = \Phi(\mathcal{H}_3)$; and if $1 \notin \Phi(\mathcal{H})$, then $\mathcal{H}_3$ can be chosen as a bi-hypergraph.*

Remark 1. *Deleting the condition $|\mathcal{C}| + |\mathcal{D}| \geq 1$ from Theorem 1 (i.e. admitting mixed hypergraphs and bi-hypergraphs without hyperedges), we obtain that S is a feasible set of an r-uniform mixed hypergraph if and only if S satisfies*

(i) or (ii) or

(iii)′ $\min(S) = 1$ *and $S = \{1, \ldots, \bar{\chi}\}$ for some natural number $\bar{\chi}$.*

The same is true for feasible sets of r-uniform bi-hypergraphs.

We begin with some easy observations, on fewer than r colors, in Section 2.2. The essential part of the proof of Theorem 1 is split into three sections. In Section 2.3 we construct bi-hypergraphs whose feasible sets contain just one element larger than $r-1$. Then in Section 2.4 we show how to combine them in order to generate a feasible set with an unrestricted number of prescribed elements larger than $r-1$, but still admitting an $(r-1)$-coloring. The feasible sets having no elements smaller than r are treated in Section 2.5.

2.2 Necessity and few colors

In this short section we prove some simple facts, implying that no other types of feasible sets can exist for any $r \geq 3$ than the ones listed in Theorem 1, and that the 'intervals' with largest element $r-1$ always are feasible. The former assertion follows directly from the next observation, taking into account that every $\Phi(\mathcal{H})$ containing 1 must be an interval, and such an $\mathcal{H}$ cannot be a bi-hypergraph containing at least one bi-edge.

Lemma 1. *Let $\mathcal{H}$ be an r-uniform mixed hypergraph with at least one hyperedge, and $k \leq r-2$ a natural number. If $\mathcal{H}$ has a k-coloring, then it also has a $(k+1)$-coloring.*

Proof Since $\mathcal{H}$ is not edgeless, it has at least r vertices. Thus, in any coloring with the colors $1, \dots, k$, two vertices get the same color. Assigning color $k+1$ to one of them, every $\mathcal{D}$-edge remains properly colored. Moreover, $k+1 \leq r-1$ still holds by assumption, hence no multicolored $\mathcal{C}$-edge can arise either. Thus, a $(k+1)$-coloring of $\mathcal{H}$ is obtained. □

The existence of feasible sets without elements larger than $r-1$ is now settled by the following claim.

Lemma 2. *Let $r \geq 3$ and $k \geq 2$ be integers, $k < r$. Assume that $(k-1)(r-1)+1 \leq |X| \leq k(r-1)$, and let both $\mathcal{C}$ and $\mathcal{D}$ consist of all the r-element subsets of X. Then the bi-hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ has a k-coloring, but it does not have any colorings with at most $k-1$ or at least r colors.*

Proof By the pigeon-hole principle, fewer than k colors would yield that some color class has at least r vertices, hence a 'forbidden' monochromatic $\mathcal{D}$-edge would occur. Whilst assuming a coloring with at least r colors, there would exist an r-element multicolored $\mathcal{C}$-edge. On the other hand, the upper bound on $|X|$ implies that X admits a partition into k classes, each of which has cardinality at most $r-1$. In this way no monochromatic $\mathcal{D}$-edge occurs, moreover no multicolored $\mathcal{C}$-edge is created either, because $k < r$. □

Since $(k-1)(r-1)+1 < k(r-1)$ holds whenever $r > 2$, there is enough room to choose $|X|$ for any r and k. Then the bi-hypergraph constructed admits a coloring with any number of colors between k and $r-1$.

From now on, in this chapter a bi-hypergraph $\mathcal{H}$ will be denoted simply by $\mathcal{H} = (X, \mathcal{E})$ instead of $\mathcal{H} = (X, \mathcal{E}, \mathcal{E})$.

2.3 Basic blocks for many colors

Let us introduce the following notation: $S' = \{i \in \mathbb{N} \mid \ell \le i \le r-1\}$, where $\ell \ge 2$ and $r \ge 3$ are fixed integers. That is, S' is the 'interval' $\{\ell, \ell+1, \dots, r-1\}$ or just $\{r-1\}$ or the empty set. The set $S'' = \{k_1, k_2, \dots, k_m\}$ contains integers satisfying $\min(S'') \ge r$. It was stated in the parts (i) and (ii) of our main theorem that, for every S' and S'' (assuming $S' \cup S'' \ne \emptyset$), there exists an r-uniform bi-hypergraph $\mathcal{H}$ complying with $\Phi(\mathcal{H}) = S' \cup S''$.

We have shown in Section 2.2 that the assertion is valid whenever $|S'| \ge 1$ and $|S''| = 0$. Now we prove it for the case of $|S'| \ge 1$ and $|S''| = 1$.

Lemma 3. *For all integers $r \ge 3$, $k \ge r$, and $2 \le \ell < r$, there exists an r-uniform bi-hypergraph $\mathcal{H}$ whose feasible set is $\{i \mid \ell \le i \le r-1\} \cup \{k\}$. Moreover, this $\mathcal{H}$ has precisely one k-coloring, apart from the renumbering of colors.*

Proof We construct a bi-hypergraph $\mathcal{H} = (X, \mathcal{E})$ with the claimed property as follows. Let the vertex set X be the union of k sets, each of them containing $r-1$ consecutive vertices:

$$B_j = \{x_{(j-1)(r-1)+1}, \dots, x_{j(r-1)}\} \qquad \text{for all} \quad 1 \le j \le k\,; \qquad X = \bigcup_{j=1}^{k} B_j$$

In the sequel, we shall refer to those B_j as *branches*, and their last elements $x_{j(r-1)}$ as *end-vertices*. Furthermore, let the *distance* of two vertices in the same branch be introduced as the difference of their indices. We emphasize that this term is not defined and applied in the case when the vertices belong to different branches. Two vertices will be termed *consecutive* if their distance is exactly 1.

An r-element subset of X is chosen to be a bi-edge of $\mathcal{H}$ if it contains two vertices having distance at most $\ell - 1$. (Consequently, these two vertices are from the same branch.) Formally:

$$\mathcal{E} = \left\{ E \mid E \in \binom{X}{r} \ \wedge\ \exists j,k \text{ s.t.} \left(x_j, x_k \in E \ \wedge\ |j-k| < \ell \ \ \wedge\ \ \left\lceil \tfrac{j}{r-1} \right\rceil = \left\lceil \tfrac{k}{r-1} \right\rceil \right) \right\}$$

Observe now the possible colorings c of the bi-hypergraph $\mathcal{H} = (X, \mathcal{E})$.

($\star$) *If there exists a non-monochromatic branch in the coloring c of $\mathcal{H}$, then $\mathcal{H}$ has at most $r-1$ colors in c.*

Assume a coloring c with at least r colors and a non-monochromatic branch. There surely exist two consecutive vertices, say a and b, with different colors in the non-monochromatic branch. Since at least r colors are used in c, one can choose $r-2$ vertices besides a and b to produce a totally multicolored r-element vertex set. But because of having a and b within distance $\ell - 1$, these r vertices would form a multicolored bi-edge in $\mathcal{H}$. This is a contradiction, therefore the number of colors is smaller than r in c.

($\star\star$) *If there exists a color occurring twice within distance $\ell - 1$ in coloring c, then $\mathcal{H}$ is colored with more than ℓ colors.*

A color or a color class will be termed *close-repeated* if it has two vertices in distance at most $\ell - 1$. Assume for a moment that a close-repeated color class has at least r elements. In this case one could choose two vertices within distance $\ell - 1$ and further $r - 2$ vertices, all of them belonging to this class. It would yield a forbidden monochromatic bi-edge, consequently a close-repeated color class has at most $r - 1$ elements.

Let us write $r - 1$ in the form $r - 1 = a\ell + b$, where $a \geq 1$ and $0 \leq b < \ell$ are integers. Now, consider a coloring c having some number $s \geq 1$ of close-repeated color classes, and altogether at most ℓ colors. By the above observation, the union of close-repeated color classes can have at most $s(r-1)$ elements. Consider the other at most $\ell - s$ remaining color classes. Since each of these colors appears at most once on any ℓ consecutive vertices, each branch has at most $a(\ell-s)+b$ vertices with those remaining colors. Regarding all the k branches we obtain that the union of those color classes contains at most $k[a(\ell - s) + b]$ vertices. Consequently, for the assumed coloring c of $\mathcal{H}$ the following inequality should hold:

$$s(r-1) + k[a(\ell - s) + b] \geq k(r-1) = k(a\ell + b)$$

But this would yield the inequality $s(r - 1) \geq kas$, what contradicts the condition $s \geq 1$ and the fact $0 < r - 1 < k \leq ak$ derived from the definition of $\mathcal{H}$. Hence, if the bi-hypergraph $\mathcal{H}$ contains a monochromatic vertex pair within distance $\ell - 1$ in c, then it necessarily has more than ℓ colors.

(1) *$\mathcal{H}$ has no coloring with fewer than ℓ colors.*

In the case of coloring $\mathcal{H}$ with at most $\ell - 1$ colors we would surely have a color repeated within distance $\ell - 1$, but according to ($\star\star$) it is impossible.

(2) *$\mathcal{H}$ is ℓ-colorable, and an ℓ-coloring is proper for $\mathcal{H}$ if and only if every ℓ consecutive vertices from the same branch have mutually different colors; that is, each branch has a periodic ℓ-coloring.*

On the one hand, by ($\star\star$) there cannot appear close-repeated colors in an ℓ-coloring of $\mathcal{H}$. On the other hand, if any two vertices within distance $\ell - 1$ have different colors, each bi-edge surely has non-monochromatic vertices. Moreover, because $\ell < r$ holds, there obviously exist some vertices with the same color. Therefore all the ℓ-colorings of the described type are appropriate for $\mathcal{H}$.

(3) *$\mathcal{H}$ is j-colorable for all $\ell < j \leq r - 1$.*

This follows immediately from Lemma 1 and the assertion (2) above.

(4) *The only coloring of $\mathcal{H}$ with more than $r-1$ colors is the k-coloring where each branch is monochromatic.*

According to ($\star$), using more than $r-1$ colors each branch is monochromatic. For any two branches there exist bi-edges contained in their union. Consequently, in order to avoid the appearance of monochromatic bi-edges, the branches have to get mutually different colors. This particular k-coloring with monochromatic branches is appropriate for all the bi-edges. Since every color class (branch) has fewer than r elements, any r-element edge surely has some vertices with different colors. Moreover, each edge has some vertices belonging to the same branch, that is, from the same color class, so it cannot be totally multicolored. Therefore $\mathcal{H}$ is k-colorable, but this is the only coloring with more than $r-1$ colors.

According to the assertions (1)–(4) above, the constructed bi-hypergraph has the prescribed feasible set $\{i \mid \ell \leq i \leq r-1\} \cup \{k\}$. □

(The terminology introduced here will be used throughout this chapter.)

2.4 Joining the components

In this section we prove Theorem 1 for $|S'| \geq 1$ and $|S''| \geq 2$.

Lemma 4. *For all integers $r \geq 3$, $m \geq 2$, $2 \leq \ell < r$ and $r \leq k_1 < k_2 < \ldots < k_m$, there exists an r-uniform bi-hypergraph $\mathcal{H}$ whose feasible set is $\Phi(\mathcal{H}) = S' \cup \{k_1, k_2, \ldots, k_m\}$, where $S' = \{i \mid \ell \leq i \leq r-1\}$.*

Proof To construct a bi-hypergraph $\mathcal{H}$ with the prescribed feasible set, we will join m mutually vertex-disjoint bi-hypergraphs $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m$ constructed by the procedure of Lemma 3 and having the following properties:

$$\mathcal{H}_i = (X_i, \mathcal{E}_i)\,, \qquad \Phi(\mathcal{H}_i) = S' \cup \{k_i\} \qquad \text{for } \ i = 1, 2, \ldots, m$$

To distinguish the vertices of different components from each other, upper indices will be used; e.g., x^i_j denotes the j-th vertex of the component $\mathcal{H}_i$. The bi-hypergraph $\mathcal{H}$ will arise by joining the components $\mathcal{H}_i$ with four new types of bi-edges:

$$\mathcal{H} = (X, \mathcal{E})\,, \qquad X = \bigcup_{i=1}^{m} X_i\,, \qquad \mathcal{E} = \mathcal{E}_\alpha \cup \mathcal{E}_\beta \cup \mathcal{E}_\gamma \cup \mathcal{E}_\delta \cup \bigcup_{i=1}^{m} \mathcal{E}_i$$

First, we describe the construction informally. The cases when there exists a component colored with more than ℓ but fewer than r colors, will be set apart from other ones. The bi-edges in $\mathcal{E}_\alpha$ force that in this case the whole $\mathcal{H}$ is colored with at most $r-1$ colors, that is $|c(\mathcal{H})| \in S'$. If there appear only ℓ- and k_i-colored components, the β-type edges force that an ℓ-colored component can be followed only by ℓ-colored ones and the number of colors in their union is smaller than r. In particular, if there is no component with more than ℓ colors, the number of colors cannot exceed $r-1$ in $\mathcal{H}$. The γ- and δ-type bi-edges will ensure that for a k_i-colored component $\mathcal{H}_i$ every preceding component $\mathcal{H}_x$ $(x < i)$ has a k_x-coloring such that $c(\mathcal{H}_x) \subset c(\mathcal{H}_i)$, moreover if a later component $\mathcal{H}_y$ $(i < y)$ has an ℓ-coloring, then $c(\mathcal{H}_y) \subset c(\mathcal{H}_i)$ holds.

We now define the $\alpha, \beta, \gamma, \delta$-type bi-edges and observe their coloring properties.

- The next α-type bi-edges appear in the r-uniform $\mathcal{H}$ only if $\min(S') = \ell < r-1$ holds.

(α) The set $\mathcal{E}_\alpha$ contains all the r-element vertex subsets E with the following properties:
— E meets at least two components,
— There exists a component $\mathcal{H}^*$ from which E contains at least $\ell+1$ vertices, two of them having distance at most $\ell - 1$, and E intersects at least two branches of $\mathcal{H}^*$.

For a particular E, we let n denote the number of vertices of E in $\mathcal{H}^*$; hence, this n ranges from $\ell + 1$ to $r - 1$ over the $\mathcal{E}_\alpha$-edges. (If $\ell + 1 \leq r/2$, then n may be multiply defined for some of the edges $E \in \mathcal{E}_\alpha$.)

(a) *If there exists a component $\mathcal{H}_i$ colored with more than ℓ but fewer than r colors, then the whole $\mathcal{H}$ has at most $r - 1$ colors.*

Consider a coloring c of $\mathcal{H}$, in which a component $\mathcal{H}_i$ has an n-coloring where $\ell + 1 \leq n \leq r - 1$. There exist at least $k_i - (n - 2) \geq 3$ non-monochromatic branches in $\mathcal{H}_i$. Choosing one of them, say B_j, there are two consecutive vertices, say a and b, with different colors. In the remaining $k_i - 1$ branches, we try to find a vertex y with a color different from $c(a)$ and $c(b)$. If there is no such vertex y, then all the $n \geq 3$ colors have to appear on B_j. Hence, starting with another non-monochromatic branch $B_j^{'}$ and its two consecutive vertices a' and b' with different colors, we can supplement them with an appropriate vertex y' from the branch B_j. In either case we obtain three vertices from exactly two branches, two of them being within distance $\ell - 1$. If $n > 3$, they can be supplemented with one vertex from each of the remaining $n - 3$ color classes of $\mathcal{H}_i$. So we have an n-element multicolored vertex set, what can be expanded to an α-bi-edge with any $r - n$ vertices of other components. Consequently, it is not possible to use $r - n$ colors in $\mathcal{H}$, each of them being different from the n colors of $\mathcal{H}_i$.

From now on we can restrict our attention to colorings where each component is either ℓ- or k_i-colored.

- The β-type bi-edges are defined as follows:

(β) An r-element vertex set E is contained in $\mathcal{E}_\beta$ if and only if there exists an index i $(1 \le i < m)$ such that
— $E \cap X_i = \{x^i_p \mid r(r-1) - \ell < p \le r(r-1)\}$, that is E contains precisely ℓ vertices from $\mathcal{H}_i$: the last ℓ vertices of the r-th branch,
— $E \cap X_j = \emptyset$ for all $j < i$,
— E does not meet the r-th branch of $\mathcal{H}_j$, if $j > i$.

(b) *If there exists an ℓ-colored component $\mathcal{H}_i$, then all the later components $\mathcal{H}_j$ $(j > i)$ have at most $r-1$ colors, moreover their union $\bigcup_{p=i}^m X_p$ has also got at most $r-1$ colors. In particular, if $\mathcal{H}_1$ has an ℓ-coloring in c, then $\ell \le |c(\mathcal{H})| \le r-1$ holds.*

Since $\mathcal{H}_i$ has a periodic ℓ-coloring, the last ℓ vertices from the r-th branch are totally multicolored. Assuming for a contradiction that a component $\mathcal{H}_j$ is k_j-colored $(i < j)$, it would have $k_j - 1$ colors outside the r-th branch. Since $r \le k_i < k_j$ holds, we obtain $r \le k_j - 1$. Therefore one can choose $r - \ell$ branches (without the r-th branch) having colors not used in $\mathcal{H}_i$. But in this case the end-vertices of these branches and the fixed ℓ vertices of $\mathcal{H}_i$ would form a forbidden multicolored bi-edge in $\mathcal{H}$.

If each of the components after $\mathcal{H}_i$ is periodically ℓ-colored, all of their colors appear on the first branches. Consequently, if at least r colors are used on the union $\bigcup_{p=i}^m X_p$, we could choose the fixed ℓ vertices from the r-th branch of $\mathcal{H}_i$, and $r - \ell$ vertices from the first branches of the later components such that they have mutually different colors. This would yield to a multicolored bi-edge, hence there are at most $r-1$ colors used after the component $\mathcal{H}_{i-1}$.

- To define the γ- and δ-type bi-edges we shall use the notation Y^i_d for the set containing the end-vertices of the 1st, 2nd,..., d-th branches of a component $\mathcal{H}_i$ (where $d \le k_i$):

$$Y^i_d = \{x^i_{p(r-1)} \mid 1 \le p \le d\}$$

(γ) The γ-bi-edges are the r-element vertex sets containing the end-vertices from the first $r-1$ branches of a component $\mathcal{H}_i$, and one more vertex from the first ℓ vertices of the first branch of any later component $\mathcal{H}_j$ $(1 \le i < j \le m)$. Formally:

$$\mathcal{E}_\gamma = \{E \mid \exists\, i, j, p \text{ s.t. } \big(E = Y^i_{r-1} \cup \{x^j_p\} \ \wedge\ 1 \le i < j \le m \ \wedge\ 1 \le p \le \ell\big)\}$$

(c) *If the component $\mathcal{H}_i$ is k_i-colored and one of the later components $\mathcal{H}_j$ $(i < j)$ is ℓ-colored, then there is no color appearing in $\mathcal{H}_j$ but not in $\mathcal{H}_i$; that is, $c(\mathcal{H}_j) \subset c(\mathcal{H}_i)$ holds.*

Considering a k_i-colored $\mathcal{H}_i$, the end-vertices in Y^i_{r-1} are totally multicolored. If the component $\mathcal{H}_j$ $(i < j)$ is ℓ-colored, because of the periodicity, all the

ℓ colors occur on its first ℓ vertices. The appearance of a multicolored bi-edge is avoidable only if all the colors of $\mathcal{H}_j$ occur also on Y^i_{r-1}, and therefore $c(\mathcal{H}_j) \subset c(\mathcal{H}_i)$.

Let us note that this case, combined with the observations (a) and (b), implies that $\mathcal{H}_p$ is ℓ-colored for all $p \geq j$, and $c\Big(\bigcup_{p=j}^{m} X_p\Big) \subset c(\mathcal{H}_i)$ holds.

- Each of the δ-type bi-edges intersects two consecutive components. We introduce them to deal with the cases when both $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ have more than $r-1$ colors.

(δ) Each bi-edge from $\mathcal{E}_\delta$ contains the first $r-2$ end-vertices of a component $\mathcal{H}_i$, and this set is supplemented for all $r-1 \leq j \leq k_i$ with the end-vertices of the j-th branches from both the $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ components. For $1 \leq j \leq r-2$ we get the δ-type edges similarly, but in this case the vertex $x^i_{j(r-1)}$ is already contained in Y^i_{r-2}, so we take the vertex $x^i_{(r-1)(r-1)}$. Hence the edge set $\mathcal{E}_\delta$ contains the edges of following forms for $1 \leq i < m$:

$$Y^i_{r-2} \cup \{x^i_{j(r-1)}, x^{i+1}_{j(r-1)}\} \qquad \text{for} \qquad r-1 \leq j \leq k_i$$

$$Y^i_{r-2} \cup \{x^i_{(r-1)(r-1)}, x^{i+1}_{j(r-1)}\} \qquad \text{for} \qquad 1 \leq j \leq r-2.$$

(d) *If $\mathcal{H}_i$ is k_i-colored and $\mathcal{H}_{i+1}$ is k_{i+1}-colored, then all the colors of $\mathcal{H}_i$ are repeated on the component $\mathcal{H}_{i+1}$, that is $c(\mathcal{H}_i) \subset c(\mathcal{H}_{i+1})$.*

Since the end-vertices of a k_i-colored component $\mathcal{H}_i$ surely have different colors, the first k_i end-vertices from the succeeding component can get no other color. Therefore, if a component $\mathcal{H}_{i+1}$ is k_{i+1}-colored ($k_i < k_{i+1}$), and consequently has end-vertices with mutually different colors, then all the k_i colors of the component $\mathcal{H}_i$ must be repeated.

According to the above assertions (a), (b), (c), and (d), only the following types of colorings can exist for $\mathcal{H}$:

- Every component has an ℓ-coloring and according to the assertion (b), $\ell \leq |c(\mathcal{H})| \leq r-1$ holds.
- There exists a component colored with at least $\ell+1$ but at most $r-1$ colors. Applying (a) we get that $\ell+1 \leq |c(\mathcal{H})| \leq r-1$ holds.
- There exist some components colored with more than $r-1$ colors. Consider such a component with the maximum index i. According to (b), for every $1 \leq x \leq i$, $\mathcal{H}_x$ is k_x-colored, and by (d) the relation $c(\mathcal{H}_x) \subset c(\mathcal{H}_i)$ holds. By the maximality of the index i, for every $i < y$ the component $\mathcal{H}_y$ is ℓ-colored and applying (c) we get that $c(\mathcal{H}_y) \subset c(\mathcal{H}_i)$ holds. These imply $|c(\mathcal{H})| = k_i$.

We have shown that $\Phi(\mathcal{H}) \subseteq S' \cup S''$. It remains to prove that there are examples of proper j-colorings for every $j \in S' \cup S''$.

Let two types of colorings be fixed in the following two examples. Let every ℓ-colored component be colored periodically with colors $1, 2, \ldots, \ell$ such that the end-vertex from the first branch has the color 2 and other end-vertices are colored with 1. In the k_i-colored components the vertices in the j-th branch all have the color j, for all $1 \leq j \leq k_i$.

(ℓ) Let all the components have the above ℓ-coloring. The α- and β-bi-edges contain two vertices within distance $\ell - 1$, hence with different colors. The γ- and δ-bi-edges contain $r - 1 \geq 2$ end-vertices from a component. One of these end-vertices is the first one with color 2, and there is another one with color 1. Obviously, every r-element bi-edge has some monochromatic vertices too, since $\ell < r$ holds. Therefore, the edges intersecting more than one component are properly colored and, clearly, the edges contained in one of the components have proper colorings, too. Consequently, the bi-hypergraph $\mathcal{H}$ is ℓ-colorable.

(k_i) Let the $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_i$ components have the prescribed $k_1, k_2, \ldots, k_i$-colorings, and all the remaining components be ℓ-colored (their union be referred as the ℓ-colored part of $\mathcal{H}$).

For an α-type bi-edge E there is a component $\mathcal{H}^*$ from which E contains vertices belonging to the same branch (say a and b) as well as vertices belonging to different branches (say a and d), whilst $|E \cap \mathcal{H}^*| \geq \ell + 1$ holds.
- If $\mathcal{H}^*$ has at least r colors: $c(a) = c(b) \neq c(d)$.
- If $\mathcal{H}^*$ has ℓ colors: $c(a) \neq c(b)$, and because E has at least $\ell + 1$ common vertices with the ℓ-colored $\mathcal{H}^*$, there must appear monochromatic vertices, too.

A β-type bi-edge E contains $\ell \geq 2$ consecutive vertices from the r-th branch of a component $\mathcal{H}_j$ and has no more vertex from the r-th branch of any other component.
- If this component is ℓ-colored ($i < j$), then the vertices in $E \cap \mathcal{H}_j$ have different colors, whilst E is contained in the ℓ-colored part of $\mathcal{H}$, implying that there are monochromatic vertices in E, too.
- If $\mathcal{H}_j$ is k_j-colored, then all the ℓ vertices from the r-th branch have the same color r. But there cannot be other vertices with color r in E, since this color can appear only on the r-th branches.

A γ-type bi-edge E has the first $r - 1 \geq 2$ end-vertices from a component $\mathcal{H}_j$ and one more vertex from the first branch of a later component.
- If $\mathcal{H}_j$ is ℓ-colored, the first and second end-vertices have different colors, and because E is contained in the ℓ-colored part of $\mathcal{H}$, there also exist vertices with the same color.
- If $\mathcal{H}_j$ is k_j-colored, the first $r-1$ end-vertices have colors $1, 2, \ldots, r-1$ (ensuring the different colors), and the remaining vertex is from the first branch B_1^p,

which is either monochromatic with color 1 or contains the colors $1, 2, \ldots, \ell$. In both cases, a repeated color appears.

A δ-type bi-edge E contains $r-1$ end-vertices from a component $\mathcal{H}_j$, including the first end-vertex. The first and any other end-vertices surely have different colors.
- If $\mathcal{H}_j$ is ℓ-colored, the edge E is contained in the ℓ-colored part of $\mathcal{H}_j$, therefore there exist monochromatic vertices in it.
- If both $\mathcal{H}_j$ and $\mathcal{H}_{j+1}$ are colored with at least r colors, then consider the end-vertex from $\mathcal{H}_{j+1}$ which is contained in E. By the construction there is an end-vertex with corresponding index from $\mathcal{H}_j$ in E, and they have the same color.
- If $j = i$, that is $\mathcal{H}_j$ is k_j-colored and $\mathcal{H}_{j+1}$ is ℓ-colored, we distinguish between two cases. Assuming that E contains the first end-vertex from $\mathcal{H}_{j+1}$, it has the color 2, the same as the color of the second end-vertex of $\mathcal{H}_j$, which is also contained in E. In the other case E contains the x-th ($x > 1$) end-vertex from $\mathcal{H}_{j+1}$ whose color is 1, like the color of the first end-vertex from $\mathcal{H}_j$.

Consequently, the described k_i-coloring is proper for $\mathcal{H}$, that is $k_i \in \Phi(\mathcal{H})$.

(n) It follows immediately from Lemma 1 and the example (ℓ) above that $\mathcal{H}$ has an n-coloring for all $\ell + 1 \leq n \leq r - 1$.

Therefore the constructed r-uniform bi-hypergraph $\mathcal{H}$ complies with $\Phi(\mathcal{H}) = S' \cup S''$. □

2.5 No colorings with few colors

To complete the proof of Theorem 1, we deal with the case where the bi-hypergraph has no coloring with fewer than r colors; that is, $|S'| = 0$. First, consider the case when S'' contains at least two integers.

Lemma 5. *For every integer $r \geq 3$ and for every set $S'' = \{k_1, k_2, \ldots, k_m\}$ of integers, where $m \geq 2$ and $r \leq k_1 < k_2 < \ldots < k_m$, there exists an r-uniform bi-hypergraph with feasible set S''.*

Proof Let us take the r-uniform bi-hypergraph $\mathcal{H}^*$ constructed according to Lemma 4 with feasible set $\{r-1\} \cup S''$, as our starting-point. The components of $\mathcal{H}^*$ are denoted by $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m$. The bi-hypergraph $\mathcal{H}$ complying with the conditions of this lemma will be obtained by supplementing $\mathcal{H}^*$ with some bi-edges. The new bi-edges meet only the components $\mathcal{H}_1$ and $\mathcal{H}_2$ as follows: Each of them contains precisely one vertex from each of the 1st, 2nd, ..., and $(r-1)$-st branches from $\mathcal{H}_1$, moreover contains the end-vertex of the first branch of $\mathcal{H}_2$ (namely the vertex x^2_{r-1}). Assuming that $\mathcal{H}_1$ and consequently all the components are $(r-1)$-colored, the branches are totally multicolored with the same $r-1$ colors. Hence one can choose a vertex with the same color as x^2_{r-1} from each of the 1st, 2nd, …, and $(r-1)$st branches of $\mathcal{H}_1$. But in this case we would get a forbidden monochromatic bi-edge, therefore neither $\mathcal{H}_1$ nor $\mathcal{H}$ can be $(r-1)$-colored. It is readily seen that the examples of k_i-colorings of $\mathcal{H}^*$, described in Section 2.4, remain appropriate for $\mathcal{H}$. □

Lemma 6. *For all integers r and k, where $3 \le r \le k$, there exists an r-uniform bi-hypergraph with feasible set $\{k\}$.*

Proof Consider the r-uniform bi-hypergraph $\mathcal{H}'$ with feasible set $\{k, k+1\}$, constructed from two components according to the proof of Lemma 5. We supplement it only with the bi-edge containing the first r end-vertices of the second component:

$$E' = \{x^2_{j(r-1)} \mid 1 \le j \le r\}$$

There exists just one $(k+1)$-coloring for $\mathcal{H}_2$, and E' would be multicolored in it. Consequently, the r-uniform bi-hypergraph $\mathcal{H}$, obtained by supplementing $\mathcal{H}'$ with the bi-edge E', is not $(k+1)$-colorable. But the k-coloring described in Lemma 5 is still appropriate for $\mathcal{H}$. Hence $\Phi(\mathcal{H}) = \{k\}$ holds. □

The lemmas of Sections 2.2 through 2.5 cover all possible cases, hence completing the proof of Theorem 1. □□□

It is worth noting that in the constructions above, we might have deleted some bi-edges and still have the same feasible set. The reason for not doing so was that we wanted to keep the proof relatively simple.

3 Uniform $\mathcal{C}$-hypergraphs with few colorings

If a mixed hypergraph contains only $\mathcal{C}$-edges (i.e., it is a $\mathcal{C}$-hypergraph) and every edge contains r vertices, then all colorings using $1, 2, \ldots, r-1$ colors trivially are proper, since polychromatic edges of size r cannot occur. Hence, in this case the question of colorability can be replaced by the one whether the r-uniform $\mathcal{C}$-hypergraph has a coloring with more than $r-1$ colors. The answer can be negative only if the hypergraph contains sufficiently many hyperedges. For instance, the upper chromatic number of an r-uniform $\mathcal{C}$-hypergraph of order n surely equals $r-1$ if all the $\binom{n}{r}$ vertex subsets of size r are chosen as hyperedges. But we can easily give a smaller example containing only $\binom{n}{r-1}$ edges, by considering only the r-element subsets incident to a fixed vertex x. Moreover, we will see that using a more inventive structure, the number of edges can be decreased substantially.

The asymptotically tight estimate, given in this chapter, solves an open problem raised in [58] and recalled in the monograph [59]. This result can also have a practical connection, since there occur many problems where the partition constraints concern groups of the same size r. In these cases, if the number of hyperedges is smaller than the lower bound in question, then we can be sure that the hypergraph is colorable using precisely r colors.

3.1 About the earlier results and their tightness

The main purpose of this chapter is to solve Problem 2 from Section 2.6 (page 43) of [59] — raised already in Problem 11 of [58] in implicit form — for the most important case when we ask for the minimum number of hyperedges in an r-uniform $\mathcal{C}$-hypergraph $\mathcal{H}$ of order n with $\overline{\chi}(\mathcal{H}) < r$. This minimum value is denoted by $f(n, r)$.

Despite that the determination of $f(n, r)$ looks quite a fundamental question, we have been able to find only very few related results, as listed below.

- For the two-variable extremal function $f(n, r)$ the lower bound

  $$f(n, r) \geq \frac{2}{n-r+2}\binom{n}{r} \qquad (1)$$

  was proved.

- For $r = 2$, one can immediately see that $f(n, 2) = n - 1$ for all $n \geq 2$, because it is the minimum number of edges in a connected graph on n vertices.

- For $r = 3$ and all $n \geq 3$, $f(n, 3) = \lceil n(n-2)/3 \rceil$.

- For $r = 4$ and $n \geq 3$, $f(n, 4) \leq n^3/8 + O(n^2)$.

But for $r = 4$, the above estimate is neither tight, nor asymptotically tight and for $r > 4$ there were known non-trivial estimates at all.

In fact, as it is stated in our Proposition 1, for each $r > 3$, there exist infinitely many values of n, for which the lower bound (1) is not tight. Actually, the difference goes to infinity as $n \to \infty$ if $r \geq 4$. But we will prove in Theorem 2 that the lower bound (1) is asymptotically tight for every fixed r, and also if $r = o(\sqrt[3]{n})$.

Proposition 1. *If* $\quad n \equiv r + 1 \pmod 3 \quad$ *then*

$$f(n,r) \geq \frac{2}{n-r+2}\binom{n}{r} + \frac{1}{3}\frac{\binom{n}{r-3}}{\binom{r}{r-3}}.$$

The proof will appear in the journal version of this work (under review).

3.2 Asymptotically tight estimate

Here we state our main theorem that asymptotically solves the problem for all fixed values of r, moreover this gives asymptotically tight estimates for all $r = o(n^{1/3})$ as $n \to \infty$. As usual, the family of all r-element subsets of a set X will be denoted by $\binom{X}{r}$.

Theorem 2. *For the minimum number $f(n,r)$ of hyperedges in an r-uniform $\mathcal{C}$-hypergraph with upper chromatic number $r-1$ the following estimates hold for all integers $n > r > 2$:*

(i) $f(n,r) \leq \frac{2}{n-1}\binom{n-1}{r} + \frac{n-1}{r-1}\left(\binom{n-2}{r-2} - \binom{n-r-1}{r-2}\right)$ *for all n and r.*

(ii) $f(n,r) = (1+o(1))\,\frac{2}{r}\binom{n-2}{r-1}$ *for all $r = o(n^{1/3})$ as $n \to \infty$.*

The proof will appear in the journal version of this work (under review).

In [9] we consider this problem in a more general setting and estimate the minimum number of $\mathcal{C}$-edges in an r-uniform $\mathcal{C}$-hypergraph $\mathcal{H}$ of order n, for which $\overline{\chi}(\mathcal{H}) < k$. The case of $k = r$, discussed in this chapter, plays central role there, since our construction can be modified to obtain upper bounds on minimum numbers for cases when $k > r$.

4 $\mathcal{C}$-perfect hypertrees

Perfect graphs[8] play a central role in the theory of graph coloring. Theoretically, this concept has been one of the driving forces for research in graph theory from the early 1960's. From an algorithmic point of view, although perfect graphs form a quite wide subclass of graphs and contains many important types, they admit efficient algorithms for many problems that are NP-complete in general.

For mixed hypergraphs, Voloshin [58] introduced the concept of $\mathcal{C}$-perfectness that can be viewed as dual of graph perfectness. Although it has not been proved for the whole class of $\mathcal{C}$-perfect mixed hypergraphs yet, there is an expectation that they admit efficient coloring algorithms, contrary to mixed hypergraphs in general.

The characterization of $\mathcal{C}$-perfect hypergraphs is still an open problem, even for some interesting particular cases. In this chapter we study a subclass of both theoretical and algorithmic importance, called $\mathcal{C}$-perfect hypertrees, with emphasis on those with $\mathcal{C}$-edges only. The notion is very simple and looks promising in connection with applications, too. Starting with a tree graph, some of its subtrees can be taken as $\mathcal{C}$-edges (in the more general case, $\mathcal{D}$-edges of this type may also occur). Already from the early years of mixed hypergraph theory, there has been a conjecture for the characterization of the $\mathcal{C}$-perfect members in the class of $\mathcal{C}$-hypertrees. The solution of this ten-year-old problem — strongly related to a polynomial-time algorithm, too — is one of the main results in this chapter. Moreover, we obtain some complexity results refuting previous expectations.

4.1 History of the problem and new results

The main result of this chapter is the proof of a conjecture raised by Voloshin in 1995. In [58], the concept of $\mathcal{C}$-perfectness was introduced and a characterization for $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees was proposed. We observe that the corresponding characterization does not hold in general for mixed hypertrees, but it holds for hypertrees under some not too restrictive conditions. In particular, the structural property conjectured for $\mathcal{C}$-hypertrees is valid. The proof is constructive and leads to a fast coloring algorithm, too.

On the other hand, a quite unexpected complexity result is given here. In spite of the concise description of the class of $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees, the corresponding recognition problem is co-NP-complete.

[8] A graph G is called perfect if, for every induced subgraph $G' \subseteq G$, the chromatic number $\chi(G')$ is equal to the clique number $\omega(G')$, that is the number of vertices in the largest complete subgraph of G'.

Examples. Voloshin [58] considered the following basic examples with respect to $\mathcal{C}$-perfectness.

- Monostars are *not* $\mathcal{C}$-perfect (see Figure 1).
- Bistars *are* $\mathcal{C}$-perfect.
- Polystars are *not* $\mathcal{C}$-perfect.
- The cycloid $\mathcal{C}_n^r$ is $\mathcal{C}$-perfect if $n \le 2r - 2$, it is inclusion-wise minimally $\mathcal{C}$-imperfect if $n = 2r - 1$ (see Figure 1), and it contains a monostar on $2r - 1$ vertices if $n \ge 2r$ and so in this case it is not $\mathcal{C}$-perfect and not minimally $\mathcal{C}$-imperfect either.

$\mathcal{C}$-perfect uniform hypergraphs. It was conjectured for some time [58] that an r-uniform $\mathcal{C}$-hypergraph is $\mathcal{C}$-perfect if and only if it contains no monostar and no cycloid $\mathcal{C}_{2r-1}^r$ as an induced subhypergraph. This has been disproved by Král' [31] who constructed one further minimally $\mathcal{C}$-imperfect $\mathcal{C}$-hypergraph for each $r \ge 3$, on $2r$ vertices. Recently, the present author has found a larger family of examples for $r \ge 4$, namely an increasing number of minimally $\mathcal{C}$-imperfect r-uniform $\mathcal{C}$-hypergraphs as r gets large. There is some hope to characterize $\mathcal{C}$-perfect r-uniform $\mathcal{C}$-hypergraphs; but the general characterization problem of $\mathcal{C}$-perfect (or that of minimally $\mathcal{C}$-imperfect) mixed hypergraphs appears to be rather hard; Proposition 3 will be an indication in this direction. In particular, it remains an open problem whether or not there are more than six 3-uniform minimally $\mathcal{C}$-imperfect $\mathcal{C}$-hypergraphs. (Four of the known examples are monostars, and the two others are the cycloid $\mathcal{C}_5^3$ and Král's construction on six vertices [31].)

$\mathcal{C}$-perfect hypertrees. Let us give a brief summary of what has been published on the $\mathcal{C}$-perfectness of mixed hypertrees.

- In [58, Theorem 4.29], it was stated that a $\mathcal{C}$-hypertree is $\mathcal{C}$-perfect if and only if it contains no monostars as *induced subhypergraphs*. The 'only if' part follows from the fact that monostars are not $\mathcal{C}$-perfect. On the other hand, it turned out later that the original argument in [58] for the 'if' part does not work.
- In [59, Theorem 5.17] it was proved that if a mixed hypertree does not contain any polystar as a *subhypergraph* — i.e., not only the induced polystars are excluded — then it is $\mathcal{C}$-perfect. In particular, if a $\mathcal{C}$-hypertree does not contain any monostar as a subhypergraph, then it is $\mathcal{C}$-perfect.
- Bulgaru and Voloshin [16] proved that a mixed interval hypergraph is perfect if and only if it has no induced polystars. This means the exclusion of monostars and 2-polystars.

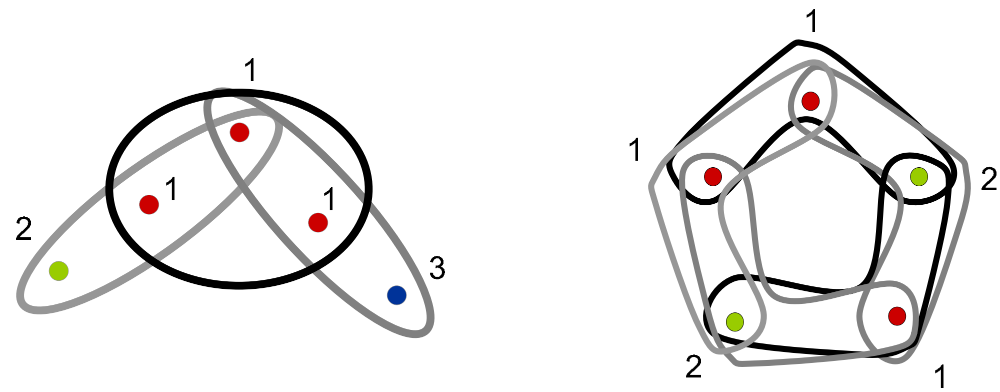

Figure 1: A 3-uniform monostar and the cycloid $\mathcal{C}_5^3$ colored with maximum number of colors. For the monostar $\bar{\chi} = 3 < \alpha_C = 4$, for the cycloid $\bar{\chi} = 2 < \alpha_C = 3$ holds. Both are minimally $\mathcal{C}$-imperfect.

New results. Our main positive result is a sufficient condition for $\mathcal{C}$-perfectness. In order to formulate it, we need to introduce the following notation. For a mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ over a host tree T, we denote

$$\mathcal{D}_2 = \{D \in \mathcal{D} : |D| = 2\}$$

that can be viewed as a subforest of T (possibly edgeless).

Theorem 3. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a colorable mixed hypertree such that all but at most one vertex has degree 0 or 1 in $\mathcal{D}_2$. If $\mathcal{H}$ contains no induced polystar, then it is $\mathcal{C}$-perfect, and a proper coloring of $\mathcal{H}$ with $\bar{\chi}(\mathcal{H})$ colors can be found in polynomial time.*

From the negative side, our main result is a rather unexpected one. In fact, a strong expectation is suggested in [59, p. 85] that $\mathcal{C}$-perfect mixed hypertrees can be recognized and $\bar{\chi}$-colored efficiently. While the latter may be true (as we prove it for the subclass described in Theorem 3), the former is refuted by the next result.

Theorem 4. *The recognition problem of $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees is* co-NP*-complete.*

We observe further that the non-hereditary version, too, of the defining property $\bar{\chi} = \alpha_C$ of $\mathcal{C}$-perfectness is hard to test. This fact is inherent in the paper [34]; it may be read out from the proofs there, but was not formulated explicitly. In paper [7] we give an independent self-contained proof.

Theorem 5. *The problem of deciding whether $\alpha_C(\mathcal{H}) = \bar{\chi}(\mathcal{H})$ is* NP*-complete over the class of $\mathcal{C}$-hypertrees.*

Returning to the positive side, in spite of the preceding results, the following constructive approach can be applied.

Theorem 6. *Over the class of colorable mixed hypertrees $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ such that all but at most one vertex has degree 0 or 1 in $\mathcal{D}_2$, there exists a polynomial-time algorithm whose output is either an induced polystar subhypergraph or a proper coloring of $\mathcal{H}$ with $\alpha_C(\mathcal{H}) = \bar{\chi}(\mathcal{H})$ colors.*

Theorem 3 has some interesting consequences. First of all, it implies that the characterization of $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees, as proposed in [58], is valid indeed.

Corollary 3. *A $\mathcal{C}$-hypertree is $\mathcal{C}$-perfect if and only if it contains no monostar as an induced subhypergraph. Moreover, $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees can be $\bar{\chi}$-colored in polynomial time.*

Also, the exclusion of 2-element $\mathcal{D}$-edges leads to a characterization.

Corollary 4. *A mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $\mathcal{D}_2 = \emptyset$ is $\mathcal{C}$-perfect if and only if it contains no monostar as an induced subhypergraph. Those $\mathcal{C}$-perfect mixed hypertrees with $\mathcal{D}_2 = \emptyset$ can be $\bar{\chi}$-colored in polynomial time.*

Remark 2. *It follows immediately by Corollary 4 that an r-uniform mixed hypertree with $r \geq 3$ is $\mathcal{C}$-perfect if and only if it contains no monostar as an induced subhypergraph. We observe that if the trivially uncolorable 2-element edges in $\mathcal{C} \cap \mathcal{D}$ are excluded, then the same characterization is valid for $r = 2$ (i.e., mixed graphs), because of the following reasons:*

(i) the upper chromatic number is equal to the number of connected components of the $\mathcal{C}$-graph; and

(ii) the following sequence of equivalences is valid: this $\mathcal{C}$-graph is not a matching plus isolated vertices $\Longleftrightarrow$ it contains a star — necessarily induced — with more than one edge $\Longleftrightarrow$ its $\mathcal{C}$-stability number is larger than the number of its connected components. □

Remark 3. *The algorithm referred to in Theorems 3, 6 and in Corollaries 3, 4 has running time $O(nm)$ in the worst case, where n and m denote the number of vertices and hyperedges, respectively.*

It is important to note that only *induced* polystars are necessary to exclude for $\mathcal{C}$-perfectness, as it is shown by the following assertion.

Proposition 2. *There exists a $\mathcal{C}$-perfect $\mathcal{C}$-hypertree containing $\mathcal{C}$-monostars as (non-induced) subhypergraphs.*

Moreover, we prove that Bulgaru and Voloshin's characterization of $\mathcal{C}$-perfectness for mixed interval hypergraphs does not extend to mixed hypertrees. Our counterexample also shows that the condition on high-degree vertices of $\mathcal{D}_2$ in Theorem 3 is best possible.

Proposition 3. *There exists a mixed hypertree that is not $\mathcal{C}$-perfect although it contains no induced polystars and has only two vertices of degree higher than 1 in $\mathcal{D}_2$.*

The proofs will appear in the journal version of this work (under review).

5 Orderings of uniquely colorable mixed hypergraphs

A subgraph admitting only one proper color partition can serve as a natural and very useful starting point when we color a graph or a hypergraph. In the class of graphs, only the complete graphs — where any two vertices are adjacent by an edge — have this nice property. Although it is not always easy to find a complete subgraph of maximum cardinality, we may be satisfied with a smaller one; and to check whether a given subgraph is complete (i.e., uniquely colorable) can be done efficiently. Moreover, in classical hypergraph coloring there occur no new types of uniquely colorable hypergraphs, hence the situation does not change fundamentally.

But in the class of mixed hypergraphs there exists a wide range of systems having only one proper color partition. The corresponding recognition problem is NP-hard [56]; and a further indication for high complexity is the fact that every colorable mixed hypergraph can appear as an induced subhypergraph of some uniquely colorable one. Consequently, such type of starting point is quite hard to find for a coloring algorithm.

In this chapter we study two subclasses of uniquely colorable mixed hypergraphs. The first of them is the class of so-called UC-orderable hypergraphs. It had been expected for several years that they could be recognized efficiently. Such a result would yield better coloring algorithms for several subclasses of mixed hypergraphs. But our theorem refutes this expectation, stating that the recognition problem of UC-orderable mixed hypergraphs is NP-complete.

After this negative result we study a more restricted subclass, namely the class of uniquely UC-orderable hypergraphs. We discuss some basic properties of them, and it is expected that they may be applicable in the design of coloring algorithms. But the time complexity of the corresponding recognition problem remains open.

5.1 Uniquely colorable mixed hypergraphs

Recall that a mixed hypergraph is termed *uniquely colorable* — UC-graph, or UC, for short — if all of its proper colorings induce the same partition into color-classes. Such hypergraphs are on the boundary between colorable and uncolorable systems. It was shown in [56] that UC-graphs have a rather unrestricted structure, and the algorithmic intractability of deciding whether a given mixed hypergraph is UC was proved, too.

Here we study two subclasses of UC-graphs. UC-orderable mixed hypergraphs — equivalently to our previous definition — have the following property: there exists an order $x_1, x_2, \ldots, x_n$ of the vertices, such that if we color the vertices in this order one by one, considering only the subhypergraph induced by $\{x_1, \ldots, x_i\}$, we have just one possible color for x_i in each step (apart from the actual choice of a new color that does not appear on $\{x_i, \ldots, x_{i-1}\}$).

Niculitsa and Voloshin proved that unique colorability and UC-orderability on *mixed hypertrees* mean the same [44]; but in general the two properties are not equivalent. The smallest example demonstrating their difference consists of two disjoint 2-element $\mathcal{C}$-edges and a $\mathcal{D}$-edge containing all the four vertices. This mixed hypergraph admits the unique color partition into two classes, but it has no UC-order.

Trivially, every UC-orderable mixed hypergraph is UC (apply the original definition to $i = n$). One might expect that the converse is simple, too: UC-orderability seems to be such a special property that it might be easy to decide whether a UC-graph has it or not. This intuition, however, is far from being correct; one of our main results, Theorem 7, states that this problem is NP-complete. Along the way, an auxiliary result — may be of interest in itself, too — is proved (Corollary 5), namely that it is NP-complete to decide whether a 3-uniform hypergraph contains a vertex subset that meets every edge in precisely one vertex.

We consider a more restricted class of UC-graphs, too. Note first that if $x_1, x_2, x_3, \ldots, x_n$ is a UC-order, then so is $x_2, x_1, x_3, \ldots, x_n$ as well, obtained by the transposition of x_1 and x_2. The mixed hypergraphs with no more UC-orders, termed uniquely UC-orderable or UUC-graphs, were introduced in [56] (cf. also Problem 3 in [59, p. 76]). The smallest UC-orderable non-UUC-graph consists of three vertices mutually joined by 2-element $\mathcal{D}$-edges, that is the simple graph K_3.

We study the color-orders belonging to the (unique) UC-orders of UUC-graphs, and completely characterize them in Theorem 8. This result shows some analogy with the paper of Bacsó, Tuza and Voloshin [13] where the *size distributions* of color partitions are characterized for the uniform UC-graphs with $\mathcal{C} = \mathcal{D}$. Both in [13] and in our theorem, the structure of mixed hypergraphs themselves is not well-described, but necessary and sufficient conditions are given for their characteristics on a higher level.

We close this section with a brief summary of complexity results on mixed hypergraphs.

Complexity of some mixed hypergraph coloring problems

- It is NP-complete to decide whether a given mixed hypergraph is colorable ([56]).

- Given $\mathcal{H}$ together with a proper coloring, it is co-NP-complete to decide whether $\mathcal{H}$ is UC ([56]). (Equivalently, deciding whether $\mathcal{H}$ admits at least one further proper coloring is NP-complete.)

- Given a UC-graph $\mathcal{H}$, it is NP-complete to decide whether $\mathcal{H}$ is UC-orderable (our Theorem 7).

- It can be decided in linear time whether a given vertex-order of $\mathcal{H}$ is a UC-order (our Proposition 4).

- Given an integer $r \geq 3$ and a sequence $n_1 \geq n_2 \geq \ldots \geq n_k \geq 1$, it can be decided in linear time whether there exists an r-uniform UC-graph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ such that $\mathcal{C} = \mathcal{D}$ and in the unique coloring of $\mathcal{H}$ the color classes have respective cardinalities $n_1, \ldots, n_k$ (from the characterization in [13]).

- Given a color-order $c_1, c_2, \ldots, c_n$, it can be decided in linear time whether there exists a UUC-graph whose unique UC-order generates the given color-order (from our characterization Theorem 8).

5.2 NP-completeness of UC-orderability

The main goal of this section is to prove the following result:

Theorem 7. *Given a uniquely colorable mixed hypergraph $\mathcal{H}$ with its coloring as an input, it is* NP*-complete to decide whether $\mathcal{H}$ has a UC-ordering.*

Before the details of the proof, let us verify first the membership of UC-orderability in NP. As a matter of fact, a polynomial-time (more precisely, quadratic) test for any fixed vertex-order can be read out from the combination of ideas presented in [54] and [56]. Here we prove a stronger (best possible) time bound, as follows.

Proposition 4. *For any $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, it is decidable in linear time whether a given vertex-order $x_1, \ldots, x_n$ on X is a UC-order.*

Proof Let us note first that the expression

$$|X| + \sum_{C \in \mathcal{C}} |C| + \sum_{D \in \mathcal{D}} |D|$$

is a lower bound on the input size. We are going to present an algorithm whose running time is proportional to this sum.

Let $X_i := \{x_1, \ldots, x_i\}$, for $i = 1, 2, \ldots, n$. Assuming that X_{i-1} has been colored, the possible colors for x_i are determined by precisely those edges of $\mathcal{H}$ that are induced by X_i, contain x_i and, moreover, are of one of the following two kinds:

- $C \in \mathcal{C}$, and all colors in $C \cap X_{i-1}$ are distinct.

- $D \in \mathcal{D}$, and all colors in $D \cap X_{i-1}$ are the same.

Such edges are called *influencing $\mathcal{C}$-edges* and *influencing $\mathcal{D}$-edges* for x_i, respectively. Denoting in general by $\varphi(Y)$ the set of colors occurring on the vertices in a set $Y \subseteq X$, the *forcing set*

$$FS(i) = \bigcap \{\varphi(C \setminus \{x_i\}) : C \text{ is an influencing } \mathcal{C}\text{-edge for } x_i\}$$

lists the colors from which x_i has to get one, while the *veto set*

$$VS(i) = \bigcup \{\varphi(D \setminus \{x_i\}) : D \text{ is an influencing } \mathcal{D}\text{-edge for } x_i\}$$

contains the colors excluded from x_i. It is readily seen that x_i is a uniquely colorable vertex if and only if either $|FS(i) \setminus VS(i)| = 1$ or there is no influencing $\mathcal{C}$-edge for x_i and $VS(i) = \varphi(X_{i-1})$ (depending on whether the uniquely determined color of x_i has appeared already in X_{i-1} or not). Hence, the heart of the matter is to generate the sets $FS(i), VS(i)$ in linear time.

The algorithm runs in two phases. First, in reverse order $x_n, x_{n-1}, \ldots, x_1$ it determines the collections $\mathcal{C}_i, \mathcal{D}_i$ of those edges whose vertex of largest subscript is x_i. Having them at hand for all i, the second phase scans X in the original order $x_1, \ldots, x_n$ and constructs a partial coloring on $X_1, X_2, \ldots$ as long as it is unique. We shall also store $|\varphi(X_{i-1})|$, that equals the number of vertices x_j $(1 \leq j \leq i)$ without influencing $\mathcal{C}$-edges. In each step $i > 1$, first the influencing edges are selected from $\mathcal{C}_i$ and $\mathcal{D}_i$ for x_i, and then it is tested whether $|FS(i) \setminus VS(i)| = 1$, or $|VS(i)| = |\varphi(X_{i-1})|$ and there is no influencing $\mathcal{C}$-edge for x_i. If none of these holds for some $i \leq n$, then the algorithm terminates with concluding that $x_1, \ldots, x_n$ is not a UC-order.

One way to proceed with this in linear time — assuming adjacency list representation, that is easily constructed from another input format if necessary — is to duplicate $\mathcal{C}$ and $\mathcal{D}$ as $\mathcal{C}'$ and $\mathcal{D}'$, the set systems that will consist of the edges actually available. While at x_i, it is checked for each edge in the list of x_i whether the edge still occurs in $\mathcal{C}'$ or $\mathcal{D}'$. If so, then the edge in question is moved from there into $\mathcal{C}_i$ or $\mathcal{D}_i$. This phase is obviously fast.

For the second phase, it is convenient to create 'dual lists', i.e. listing for each edge the vertices contained in it. It will then take just $O(|H|)$ steps for any $H \in \mathcal{C}_i \cup \mathcal{D}_i$ to test whether H is influencing for x_i ; and if so, then the corresponding colors will be inserted into $FS(i)$ or $VS(i)$. After that, the color of x_i is easily determined, always taking for new color the smallest positive integer still available. (The colors assigned are conveniently stored in a block of size n.) $\square$

5.2.1 Structure of the NP-hardness proof

We now turn to the substantial part of Theorem 7, that is the hardness of deciding whether an input mixed hypergraph admits a UC-order. The complexity of this problem will be traced back to the classical problem of *hypergraph 2-coloring*, more precisely to the 2-colorability of 3-uniform hypergraphs. For the latter, the input is a hypergraph (i.e., $\mathcal{D}$-hypergraph in the terminology of mixed hypergraphs) in which each hyperedge contains precisely 3 vertices, and the question is whether there exists a vertex partition into two classes, none of them containing any hyperedge. This problem is well-known to be NP-complete (Lovász [38]).

The reduction will be carried out in two steps. First we make a reduction from hypergraph colorability to a new type of hypergraph covering problem (still no $\mathcal{C}$-edges are involved), and then go on to UC-orderings. The proof of the latter will be postponed to the next subsection. We begin with introducing the following concept.

Definition. Let $\mathcal{F}$ be a set system over an underlying set X. A set $B \subset X$ is a *strong blocking set* (SBS, for short) if it contains precisely one element from each member of $\mathcal{F}$; that is, $|B \cap F| = 1$ holds for all $F \in \mathcal{F}$. (This term is borrowed from design theory, where 'blocking set' means a set B that meets all the $F \in \mathcal{F}$ but does not contain any of them.)

We shall prove the following lemmas. The technical conditions included in the first one will play a role in the proof of the main result later.

Lemma 7. *For any given 3-uniform hypergraph $\mathcal{E}$, a 3-uniform hypergraph $\mathcal{F}$ can be constructed in polynomial time, with the following properties: $\mathcal{E}$ is colorable with two colors if and only if $\mathcal{F}$ has a strong blocking set, moreover*

(*i*) *$\mathcal{F}$ has no blocking vertex (that is, $\bigcap_{F\in\mathcal{F}} F = \emptyset$),*

(*ii*) *$\mathcal{F}$ contains two vertices not belonging to a common hyperedge.*

Lemma 8. *Let $\mathcal{F}$ be any hypergraph with at least two vertices not belonging to a common hyperedge. Then a uniquely colorable mixed hypergraph $\mathcal{H}$ (whose coloring is known) can be constructed in polynomial time, such that $\mathcal{H}$ has a UC-order if and only if $\mathcal{F}$ has a strong blocking set with more than one element.*

¿From these two assertions, the main result of the section can easily be deduced.

Proof of Theorem 1 A 1-element strong blocking set in a hypergraph $\mathcal{F}$ would be a blocking vertex of $\mathcal{F}$. Therefore, combining Lemmas 1 and 2, we obtain that for each 3-uniform hypergraph $\mathcal{E}$, a uniquely colorable mixed hypergraph $\mathcal{H}$ (with its known coloring) can be constructed in polynomial time, such that $\mathcal{E}$ is 2-colorable if and only if $\mathcal{H}$ has a uniquely colorable ordering. Since the former property is NP-complete to decide [38], it follows that the latter is intractable, too. □

Stopping at half way, from Lemma 1 we obtain:

Corollary 5. *It is NP-complete to decide whether a 3-uniform hypergraph has a strong blocking set.* □

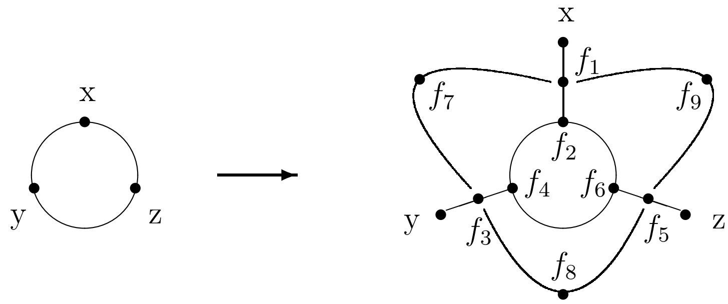

Figure 2: Each edge $E = \{x, y, z\}$ of $\mathcal{E}$ is replaced by a twelve-vertex 'gadget' in $\mathcal{F}$, constructed as shown in this figure.

Proof of Lemma 7 Let $\mathcal{E}$ be a 3-uniform hypergraph. To construct $\mathcal{F}$, we keep the ('old') vertices of $\mathcal{E}$ and supplement them with nine new vertices for each edge. If $E = \{x, y, z\}$ is an edge of $\mathcal{E}$ and the new E-vertices in $\mathcal{F}$ are $f_1, f_2, \ldots, f_9$, then the edges of $\mathcal{F}$ are the triples $\{x, f_1, f_2\}$, $\{y, f_3, f_4\}$, $\{z, f_5, f_6\}$, $\{f_2, f_4, f_6\}$, $\{f_1, f_3, f_7\}$, $\{f_3, f_5, f_8\}$, $\{f_1, f_5, f_9\}$ (see *Figure 2* for illustration). Note that the 'old' edge E is not included anymore. Then, $\mathcal{F}$ is constructed from such gadgets that meet only at the 'old' vertices and are mutually vertex-disjoint outside.

First, we assume that $\mathcal{E}$ has a proper coloring with two colors. Let us consider the (old) vertices of the first color in $\mathcal{E}$. If the edge $E \in \mathcal{E}$ has two vertices with this color, say x and y, then, in $\mathcal{F}$ we put the vertices x, y, f_6, f_7, f_8, f_9 into the SBS to be constructed. If E has only one vertex with this color, say x, we can choose x, f_3, f_6, f_9 from this gadget into the SBS. We apply this method for all the edges of $\mathcal{E}$ and finally obtain a SBS containing exactly one vertex from each edge of $\mathcal{F}$.

Second, we prove that there exists a 2-coloring of $\mathcal{E}$ whenever $\mathcal{F}$ has a SBS (denoted B). For any gadget in $\mathcal{F}$, it is impossible that all the three of its 'old' vertices x, y, z belong to B at the same time. Indeed, otherwise none of the connected f_2, f_4, f_6 could belong to B, and it would be in contradiction to the assumption that B meets the edge $\{f_2, f_4, f_6\}$. Similarly, it cannot be the case that B does not contain any of the 'old' vertices of the gadget. Therefore we can find just one or two 'old' vertices belonging to B in each gadget. Let these vertices be colored in $\mathcal{E}$ with the first color, and the other vertices with the second color. This is a proper 2-coloring since every $\mathcal{E}$-edge has two vertices with distinct colors.

Clearly, the construction can be carried out in linear time and $\mathcal{F}$ complies with the restrictions (i) and (ii). □

Remark 4. *If the hypergraph $\mathcal{F}$ constructed in the proof has some strong blocking set T, then $|T| \geq 3|\mathcal{E}|$, because in each gadget, the four edges disjoint from $\{x, y, z\}$ cannot be covered with fewer than three vertices. (A more careful analysis yields the* *lower bound $3|\mathcal{E}| + \tau(\mathcal{E})$, where $\tau(\mathcal{E})$ is the smallest number of vertices in a set that meets all edges of $\mathcal{E}$. Later on we shall only use the fact that $\tau(\mathcal{F}) > 1$.)*

5.2.2 Strong blocking sets vs. UC-orders

Here we prove Lemma 8 . We shall need the following definition.

Definition. For a hypergraph $\mathcal{H}$ with vertex set X and with edges $H_1, H_2, \ldots, H_k$, an *edge-crossing set* is a subset $B \subset X$ such that $|B \cap H_i| \le 1$ for each i, $1 \le i \le k$. By definition, a set is a SBS if and only if it is edge-crossing and also meets all edges.

Construction of $\mathcal{H}$ for Lemma 8 Let the vertex-set of $\mathcal{H}$ be the disjoint union of the following sets:

- $X = \{x_1, \ldots, x_m\}$: it has $m = |\mathcal{F}|$ elements, one for each edge $F_i \in \mathcal{F}$;
- $Y = \{y_1, \ldots, y_n\}$ and $Y^* = \{y_1^*, \ldots, y_n^*\}$: these are two copies of the vertex set $\{v_1, \ldots, v_n\}$ of $\mathcal{F}$;
- $\{w; w^*\}$: two further vertices.

All the $\mathcal{D}$-edges of $\mathcal{H}$ will have just two vertices. Vaguely speaking, their induced subgraph on $X \cup Y \cup \{w\}$ will be nearly complete (but if v_i and v_j are contained in a common edge in $\mathcal{F}$ then the $\mathcal{D}$-edge $\{y_i, y_j\}$ is missing in $\mathcal{H}$), and complete-bipartite minus a perfect matching between Y and Y^*. Formally,

$$\begin{aligned}
\mathcal{D}(\mathcal{H}) \quad = \quad & \{\{x_i, x_j\} \mid 1 \le i < j \le m\} \\
& \bigcup \{\{x_i, y_j\} \mid x_i \in X \wedge y_j \in Y\} \\
& \bigcup \{\{w, z\} \mid z \in X \cup Y\} \\
& \bigcup \{\{y_i, y_j\} \mid y_i, y_j \in Y \wedge \nexists F_k \in \mathcal{F} : (\{v_i, v_j\} \subset F_k)\} \\
& \bigcup \{\{w^*, y_i^*\} \mid y_i^* \in Y^*\} \\
& \bigcup \{\{y_i, y_j^*\} \mid y_i \in Y \wedge y_j^* \in Y^* \wedge i \ne j\}
\end{aligned}$$

The $\mathcal{C}$-edges of $\mathcal{H}$ are of two types:

- $\mathcal{C}$-Type-1 edges: $C_i^1 = \{w^*, y_i, y_i^*\}$, for every $1 \le i \le n$.
- $\mathcal{C}$-Type-2 edges: $C_i^2 = \{w, w^*, y_i\} \bigcup \{x_j \in X \mid v_i \notin F_j\}$ for every $1 \le i \le n$, where x_j corresponds to the edge F_j of $\mathcal{F}$ and y_i is the copy of the vertex v_i of $\mathcal{F}$. So, y_i and x_j belong to a common $\mathcal{C}$-edge in $\mathcal{H}$ if and only if the i-th vertex is not an element of the j-th edge in $\mathcal{F}$.

$$\mathcal{C}(\mathcal{H}) = \{C_i^1, C_i^2 \mid 1 \le i \le n\}$$

It is clear that the construction of $\mathcal{H}$ from $\mathcal{F}$ can be carried out in polynomial time.

Unique coloring First, let us observe that the mixed hypergraph $\mathcal{H}$ is colorable. Obviously, the partition where $\{y_1, y_1^*\}, \ldots, \{y_n, y_n^*\}$, and $\{w, w^*\}$ are 2-element classes and all the other classes are singletons, is a proper coloring of $\mathcal{H}$. It will be shown that for any $\mathcal{F}$ satisfying the condition of Lemma 2, the mixed hypergraph $\mathcal{H}$ constructed from $\mathcal{F}$ in the way described above, this is the only suitable coloring of $\mathcal{H}$; that is, $\mathcal{H}$ is UC in any case. Moreover, we shall prove that $\mathcal{H}$ is UC-orderable if and only if $\mathcal{F}$ has a SBS.

Let us consider an edge-crossing set B in $\mathcal{F}$, which contains at least two vertices. (Due to the assumptions of Lemma 2, such an edge-crossing set exists.) Passing on to the hypergraph $\mathcal{H}$, let B' be the subset of Y with the elements corresponding to the vertices of B. Since B is edge-crossing, no edge of $\mathcal{F}$ can contain more than one vertex of B ; hence, any two elements of B' are surely joined by a $\mathcal{D}$-edge in $\mathcal{H}$.

First, we prove that $\mathcal{H}$ is uniquely colorable in any construction, and then search for a UC-ordering of $\mathcal{H}$ if the above B is a SBS.

Step 1 : $X \cup \{w\} \cup B'$ is complete in $\mathcal{D}$-edges, so its vertices all have different colors. Their coloring is unique in any order.

Step 2 : To color w^*, let us consider all the $\mathcal{C}$-Type-2 edges C_i^2 belonging to the members of B'. All elements of their union, except w^*, have been colored in Step 1 with mutually distinct colors. Therefore, these $\mathcal{C}$-edges can be colored properly only if w^* gets the color of some common vertex. This cannot be from Y because there are at least two elements in B', and hence none of the corresponding vertices of Y belong to the intersection of their $\mathcal{C}$-Type-2 edges. Suppose that w^* gets the color of some $x_k \in \bigcap_{y_i \in B'} C_i^2$. Choosing a vertex $v_\ell \in F_k$, we have $x_k \notin C_\ell^2$, thus all vertices of C_ℓ^2 would have different colors, what is forbidden. Thereby in every $\mathcal{H}$ constructed from any $\mathcal{F}$, the color of w and w^* must be the same.

Step 3 : Let B^* be the subset of Y^* corresponding to the elements of B. For every $y_i^* \in B^*$ we have edges

$\mathcal{C}$-Type-1 : $C_i^1 = \{y_i, y_i^*, w^*\}$

$\mathcal{D}$-edge : $\{y_i^*, w^*\}$

Since w^* has got a color different from y_i in Step 2, the only chance to color properly the edge C_i^1 is that y_i^* gets a common color with y_i.

Step 4 : In this step we color the vertices of $Y \setminus B'$ and $Y^* \setminus B^*$. First, we take a $y_l \in Y \setminus B'$. It has to be colored differently from w, from the elements of X, and from the colored elements of Y^* (which are colored like the corresponding elements of Y). Therefore, we can assign only a totally new color to y_l. Then, looking at the influencing edge C_l^1, the vertex y_l^* must be colored like y_l. Thus, taking the pairs y_l, y_l^* one by one, each of them turns out to be monochromatic in a color different from all preceding colors.

UC-order from SBS Suppose that B is a SBS in $\mathcal{F}$ with at least two elements; say, $B = \{v_1, v_2, \ldots, v_k\}$. Then we can construct the following UC-order of $\mathcal{H}$:

$$x_1, \ldots, x_m \,;\; y_1, \ldots, y_k \,;\; w, w^* \,;\; y_1^*, \ldots, y_k^* \,;\; y_{k+1}, y_{k+1}^*, \ldots, y_n, y_n^* \,.$$

Until w, we obtain a heterochromatic color-order, by Step 1. The crucial point is that now Step 2 applies to w^*, even if we disregard the later vertices of $\mathcal{H}$, because B meets *all* edges of $\mathcal{F}$ — implying that the intersection of the corresponding $\mathcal{C}$-edges is empty inside X — and therefore w^* cannot get any color from X. Thus, w and w^* must have a common color, and after that the vertex-order remains UC, by Steps 3 and 4.

This argument already indicates the substantial difference between an edge-crossing set and a strong blocking set with respect to UC-orders. At the moment when both w and w^* are present in the subsequence (whichever comes later), it should be verified that they must get a common color. For this purpose, an edge-crossing set is insufficient if it fails to be a SBS.

SBS from UC-order We have already seen that every $\mathcal{H}$ obtained by the $\mathcal{F} \to \mathcal{H}$ construction is uniquely colorable, with well-defined monochromatic pairs of vertices. Suppose that $\mathcal{H}$ is not only UC but also admits a UC-order. We concentrate on the subsequence where the very first monochromatic pair appears. By what has been said, only the following possibilities may occur:

1. w repeats the color of w^*
2. some y_i repeats the color of y_i^*
3. some y_i^* repeats the color of y_i
4. w^* repeats the color of w

We are going to prove that the first three of these cannot be the case in a UC-order; and if the fourth one does, then it also results in a SBS of $\mathcal{F}$. Note that any UC-order (if it exists) has to satisfy the following requirement:

($\star$) The vertices preceding the occurrence of the first repeated color must induce a complete $\mathcal{D}$-graph. In particular, up to that point there are no colored pairs (y_i, y_i^*), and at most one y_i^* may occur.

1. If the first repeated color is at w, then w^* has been colored before coloring w. Since all $\mathcal{D}$-edges incident with w^* have their other endpoint in Y^*, ($\star$) implies that w is preceded either by w^* alone or by w^* and just one y_i^*. Hence, the subsequence ending with w does not induce any $\mathcal{C}$-edges, therefore nothing can force w to get a common color with w^*.

2. The unique coloring of y_i with the first repeated color requires some $\mathcal{C}$-edge containing y_i. Since every $\mathcal{C}$-edge involves w^*, this case can occur only if w^* and y_i^* have been colored before y_i. But with the presence of w^* we obtain the same

situation as in Case 1: Only y_i^* and w^* are colored before y_i, so y_i may have a common color with w^* instead of y_i^*.

3. Assuming, that the first color repetition is at y_i^*, the vertices y_i and w^* of the influencing $\mathcal{C}$-edge must be previously colored. But this is in contradiction to the requirement $(\star)$ since y_i and w^* are not joined by a $\mathcal{D}$-edge.

4. This is the only possible case: we have the first repeated color at w^*. Then w already appeared, and $(\star)$ implies that the vertices colored before w^* induce a complete $\mathcal{D}$-subgraph inside $X \cup Y \cup \{w\}$.

Let B' be the set of elements in Y that were colored before w^*. As they are joined by $\mathcal{D}$-edges in $\mathcal{H}$, no $F_i \in \mathcal{F}$ contains more than one of them; that is, the corresponding B in $\mathcal{F}$ is an edge-crossing set. Furthermore, the color of w^* is uniquely determined only if $|B'| \geq 2$ and the $\mathcal{C}$-Type-2 edges belonging to the $y_j \in B'$ do not contain any common element $x_i \in X$. (Otherwise at this point of the sequence w^* could get the color of w or x_i as well.) So, under the assumption that we have a UC-order, for every x_i there exists a $y_j \in B'$ such that x_i does not belong to the $\mathcal{C}$-Type-2 edge of y_j. Passing over to the hypergraph $\mathcal{F}$, for every edge F_i there is a vertex $v_j \in B$, which is contained in F_i. Thus, the edge-crossing set B meets all edges of $\mathcal{F}$, so that it is a SBS. □

5.3 Uniquely UC-orderable hypergraphs

In this section we will investigate the structure of mixed hypergraphs that have exactly one UC-order, disregarding the transposition of the first two vertices. It may be noted in general that if an edge is a subset of another edge of the same type (both are $\mathcal{C}$-edges or both are $\mathcal{D}$-edges), then the larger edge is *redundant* with respect to coloring, because it does not impose any new condition: any proper coloring for the smaller edge properly colors the larger one, too.

If the number n of vertices is at most 2, then the properties UC, UC-orderable, and UUC are equivalent. Hence, in order to avoid the few trivial exceptions, we shall assume $n \geq 3$ throughout this section. The first assertion is immediate by definition.

Proposition 5. *If $x_1, x_2, \ldots, x_n$ is the UC-order of a UUC-graph $\mathcal{H}$, then the subhypergraph of $\mathcal{H}$ induced by $\{x_j : 1 \leq j \leq i\}$ is UUC for every $i \leq n$.* □

Proposition 6. *If $\mathcal{H}$ is a UUC-graph with UC-order $x_1, \ldots, x_n$ on $n \geq 3$ vertices, then $\{x_1, x_2\}$ is a $\mathcal{C}$-edge and $\{x_1, x_2, x_3\}$ is a $\mathcal{D}$-edge. So, the subhypergraph induced by the first three vertices of the UC-order is the same in every UUC-graph without redundant edges.*

Proof By definition, x_1, x_2, x_3 is a UC-order if and only if the subhypergraphs induced by $\{x_1, x_2\}$ and by $\{x_1, x_2, x_3\}$ are uniquely colorable. Because of the uniqueness of this UC-order neither $\{x_1, x_3\}$ nor $\{x_2, x_3\}$ can be UC. Consequently, there exists an edge $\{x_1, x_2\}$, but no other 2-element edge inside $\{x_1, x_2, x_3\}$. To color x_3, we need an influencing edge for it. If $\{x_1, x_2\} \in \mathcal{D}$, the influencing edge could be $\{x_1, x_2, x_3\} \in \mathcal{C}$, but this — without a 2-element $\mathcal{D}$-edge containing x_3 — does not determine the color of x_3 uniquely. In the other case: If $\{x_1, x_2\} \in \mathcal{C}$, the influencing edge for x_3 surely is the $\mathcal{D}$-edge $\{x_1, x_2, x_3\}$. This is the only structure without redundant edges that yields a UUC subhypergraph. Note, that permitting the presence of redundant edges, this hypergraph can be supplemented with the $\mathcal{C}$-edge containing all the three vertices. □

We distinguish three types of vertices in a UC-order, depending on their colors:

- x_i has a *continuing* color if it is the same as the color of the preceding vertex x_{i-1}.
- x_i has a *returning* color if it is not continuing but this color has already occurred at some x_j $(j < i-1)$.
- x_i has a *new* color if this color has not occurred up to this point, at any x_j with $j < i$.

Accordingly, a *vertex* will be called *continuing / returning / new* if so is its color.

Proposition 7. *If $n \geq 3$, then there are no two consecutive new vertices in a UUC-order.*

Proof Due to Proposition 6, x_2 is a continuing vertex, so that the assertion is valid within $\{x_1, x_2, x_3\}$. Assuming that both x_i and x_{i+1} have new colors, for some $i \geq 3$, their positions could be switched, because of the following facts. There are $\mathcal{D}$-edges guaranteeing that the color of x_{i+1} is different from each color used up to x_{i-1}. These influencing edges cannot contain x_i. Consequently, x_{i+1} can be uniquely colored right after x_{i-1}, and then $\{x_1, \ldots, x_{i-1}, x_{i+1}, x_i\}$ also determines a UC-order of the induced subhypergraph. This cannot occur if $\mathcal{H}$ is a UUC-graph. □

For a given UUC-graph $\mathcal{H}$, we consider its unique UC-order $x_1, x_2, \ldots, x_n$. The coloring of $\mathcal{H}$ is the function c that assigns a positive integer to each x_i: $c(x_i) = c_i$. In order to associate precisely one sequence $c_1, \ldots, c_n$ of colors with the coloring c, we shall assume that the new colors $1, 2, \ldots$ appear in increasing order, without skipping any intermediate values. This $c_1, c_2, \ldots, c_n$ will be called the *color-order* of $\mathcal{H}$. If $\mathcal{H}$ is UUC, then it has one well-defined color-order determined by its unique (vertex) UC-order.

Let us summarize the necessary conditions obtained for the color-orders of UUC-graphs on $n \geq 3$ vertices.

C_0 : (By assumption) The natural numbers appear in increasing order and without gaps; i.e., $c_1 = 1$, and $1 \leq c_i \leq \max_{k<i}\{c_k\} + 1$ holds for each $2 \leq i \leq n$.

C_1 : (According to Proposition 6) $c_1 = c_2 = 1$ and $c_3 = 2$.

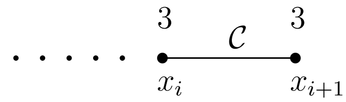

Figure 3: The only new edge when we insert the vertex x_{i+1} with the continuing color 3.

C_2 : (According to Proposition 7) If $c_i \neq c_{i+1}$, then at least one of them has occurred in the subsequence $c_1, c_2, \ldots, c_{i-1}$.

The main result of this section claims that these conditions are sufficient, too. At the end of this section, we shall compare the construction given in the proof with all UUC-graphs of the same color sequence, and prove that it is minimal from several aspects. (The hypergraph $\mathcal{H}$ constructed here will be denoted by $\mathcal{H}^*$ in the last section.)

Theorem 8. *A sequence $c_1, c_2, \ldots, c_n$ of positive integers with property C_0 is the color-order of some UUC-graph on $n \geq 3$ vertices if and only if it satisfies the requirements C_1, and C_2.*

Proof Necessity has already been shown. To prove sufficiency, we construct a suitable mixed hypergraph $\mathcal{H}$ for any given color sequence satisfying C_0, C_1, C_2.

For a mixed hypergraph $\mathcal{H}$ and the fixed vertex-order $x_1, \ldots, x_n$, we shall denote by $\mathcal{H}_i$ its subhypergraph induced by $\{x_j : 1 \leq j \leq i\}$ $(i = 1, 2, \ldots, n)$. Whenever $\mathcal{H}$ is UUC, the colors and the edges in its $\mathcal{H}_3$ have been determined above:

Colors: $c(x_1) = c(x_2) = 1$; $c(x_3) = 2$
Edges: $\{x_1, x_2\} \in \mathcal{C}$; $\{x_1, x_2, x_3\} \in \mathcal{D}$

For each $i \geq 3$ we extend $\mathcal{H}_i$ with the vertex x_{i+1} and with the following new edges:

- If c_{i+1} is a continuing color, the only new edge is: $\{x_i, x_{i+1}\} \in \mathcal{C}$. (See Fig. 3.)
- If c_{i+1} is a returning color and x_j is the last vertex before x_{i+1} with the same color as x_{i+1} (i.e., $c_{i+1} = c_j$, $j < i$, and $c_k \neq c_{i+1}$ holds for every $j < k < i+1$), the new edges are: $\{x_j, x_i, x_{i+1}\} \in \mathcal{C}$ and $\{x_i, x_{i+1}\} \in \mathcal{D}$. (See Fig. 4.)
- If c_{i+1} is a new color, we need $\mathcal{D}$-edges to distinguish the color of x_{i+1} from the previously used colors. For every color d $(d < c_{i+1})$ let $x_{d/i+1}$ denote the latest vertex colored with d before x_{i+1}. The new edges are: $\{x_{d/i+1}, x_{i+1}\} \in \mathcal{D}$, for $d = 1, \ldots, c_{i+1} - 1$. (See Fig. 5.)

It is clear that $x_1, \ldots, x_n$ is a UC-order of $\mathcal{H}$ with the given color-order $c_1, \ldots, c_n$, because the newly inserted edges and condition C_0 force x_{i+1} to get color c_{i+1} in each step.

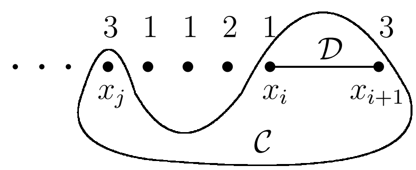

Figure 4: The new edges when we insert the vertex x_{i+1} with the returning color 3.

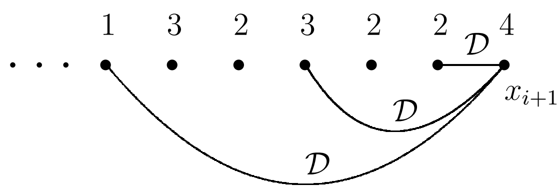

Figure 5: New edges when we insert the vertex x_{i+1} with the new color 4.

It remains to be proved that the constructed $\mathcal{H}$ has this unique UC-order only. We prove by induction on i that every subhypergraph $\mathcal{H}_i$ (induced by $\{x_1, x_2, \ldots, x_i\}$) is UUC.

It was shown that $\mathcal{H}_3$ is UUC. Suppose that $\mathcal{H}_i$ is UUC, with $x_1, x_2, \ldots, x_i$ as its only UC-order. The proof that the same holds true for $\mathcal{H}_{i+1}$ consists of two parts.

1. Deleting x_{i+1} from any UC-order of $\mathcal{H}_{i+1}$, the (unique) UC-order of $\mathcal{H}_i$ is obtained.

2. If the deletion of x_{i+1} from an UC-order of $\mathcal{H}_{i+1}$ results in the vertex-order $x_1, \ldots, x_i$ of $\mathcal{H}_i$, then x_{i+1} must be in the last position in the UC-order of $\mathcal{H}_{i+1}$.

The combination of these two assertions completes the proof of the theorem by induction.

Remark 5. *A vertex-order $y_1, y_2, \ldots, y_n$ is non-UC if and only if there exist two vertices y_k, y_j ($k < j$), such that the subhypergraph induced by $\{y_1, y_2, \ldots, y_j\}$ has two proper colorings c', c'' with $c'(y_j) = c'(y_k)$ and $c''(y_j) \neq c''(y_k)$.*

Proof of 1 Assume that the original vertex-order $x_1, x_2, \ldots, x_i$ of $\mathcal{H}_i$ is mixed and we have a different sequence, say $y_1, y_2, \ldots, y_i$. It is not a UC-order, so there is a smallest j $(1 < j \leq i)$, for which the subhypergraph $\mathcal{H}_j^{'}$ induced by $Y_j = \{y_1, y_2, \ldots y_j\}$ is not uniquely colorable. Thus, there exists y_k $(k < j)$ and a further possible coloring c^* of $\mathcal{H}_j^{'}$ beside the one determined by c, such that y_k and y_j have the same color in one of c and c^*, and different colors in the other.

Suppose for a contradiction that the deletion of x_{i+1} from a UC-order of $\mathcal{H}_{i+1}$ yields the above sequence $y_1, y_2, \ldots, y_i$. Then $\mathcal{H}_j^{'}$ with vertices $y_1, y_2, \ldots, y_j$ is not UC, so it cannot be the starting sequence of the UC-order. Therefore, x_{i+1} would have to appear earlier than y_j. We investigate $\mathcal{H}_j^{'} \cup \{x_{i+1}\}$ and prove that it cannot be UC, either.

Case 1 : The color of x_{i+1} is a continuing one in the original sequence.

If $x_i \in Y_j$, then let $c(x_{i+1}) = c(x_i)$, $c^*(x_{i+1}) = c^*(x_i)$. These are suitable colorings, and the coloring of x_{i+1} has no influence on the colors of $y_1, y_2, \ldots, y_i$. Thus, we have two different colorings, so $\mathcal{H}_j^{'} \cup \{x_{i+1}\}$ is not UC. On the other hand, if $x_i \notin Y_j$, there is no edge containing x_{i+1} in the subhypergraph so we can choose $c(x_{i+1})$ and $c^*(x_{i+1})$ as a totally new color. Therefore the extended c and c^* remain different colorings and $\mathcal{H}_j^{'} \cup \{x_{i+1}\}$ is not UC.

Case 2 : The color of x_{i+1} is originally new.

We can assign a totally new color to x_{i+1} in c^*, too. It makes no influence on the coloring of $y_1, y_2, \ldots, y_j$; thus, we have got two different colorings for $\mathcal{H}_j^{'} \cup \{x_{i+1}\}$.

Case 3 : x_{i+1} has a returning color in the original ordering.

There are at most two influencing edges for x_{i+1}, namely $\{x_l, x_i, x_{i+1}\} \in \mathcal{C}$ and $\{x_i, x_{i+1}\} \in \mathcal{D}$. By the construction, the colors of x_l and x_i are different in the original coloring c. If $x_l \notin Y_j$ or $x_i \notin Y_j$, then the $\mathcal{C}$-edge is not effective, so we let x_{i+1} have a totally new color, and then y_j still can have two different colors. Consequently, $\mathcal{H}_j' \cup \{x_{i+1}\}$ is not UC.

If x_l and x_i both are in Y_j and their colors are different not only by c but also by c^*, then x_{i+1} can get the same color as x_l, and the subhypergraph does not become UC.

The only nontrivial case is when x_l and x_i have different colors in c and the same color in c^*. In the former, x_{i+1} still can get $c(x_l)$, while in the latter it can get a totally new color. In this way we have two different proper colorings, therefore $\mathcal{H}_j' \cup \{x_{i+1}\}$ cannot be UC.

Proof of 2 We assume, from now on, that $\mathcal{H}_i$ is in its original order, $x_1, \ldots, x_i$, and x_{i+1} is inserted in a way that the sequence remains a UC-order. We need to prove that x_{i+1} is the last one.

Case 1 : x_{i+1} has a continuing color.

The only influencing edge for x_{i+1} is $\{x_i, x_{i+1}\} \in \mathcal{C}$. If x_{i+1} occurs in the sequence earlier than x_i, then in $\mathcal{H}_{i+1} - x_i$ nothing prevents x_{i+1} from getting the color $c(x_1)$, or a color different from $c(x_1)$. Thus, the vertex-order is not UC unless x_{i+1} is set at the end. As it follows, $\mathcal{H}_{i+1}$ has only one uc-ordering, that means it is a UUC.

Case 2 : x_{i+1} has a returning color.

Each edge containing x_{i+1} also contains x_i. Similarly to the previous case, it will be a UC-order only if x_{i+1} is set at the end of the sequence.

Case 3 : x_{i+1} has a new color.

There is an edge $\{x_i; x_{i+1}\} \in \mathcal{D}$, but we have no other $\mathcal{D}$-edge containing x_{i+1} and a vertex colored with $c(x_i)$. Let X_i denote the set of vertices x_j with color $c(x_i)$ but smaller subscript, $j < i$. According to condition C_2, $c(x_i)$ is not new, so that $X_i \neq \emptyset$. Since there is no edge containing both x_{i+1} and any element of X_i, it is our free choice to put x_{i+1} into the color class of X_i or assign it a distinct color, in the subhypergraph $\mathcal{H}_{i+1} - x_i$. Thus, the vertex-order is UC only if x_{i+1} is set at the end, after x_i. □

In the proof only one UUC-graph was constructed for each feasible color-order. But in fact there are a lot of UUC-graphs with different structures. Now, we list some types of measure for comparison, under which the construction of the previous proof is minimal. They show that this construction results in the simplest structure for UUC in several aspects. Assuming that the hypergraph $\mathcal{H}$ has the unique UC-order $x_1, x_2, \ldots, x_n$, we introduce the following concepts:

- *The number of edges:* $N(\mathcal{H}) = |\mathcal{H}| = |\mathcal{C}| + |\mathcal{D}| = \sum_{H \in \mathcal{H}} 1.$

- *Edge-size sum:* $S(\mathcal{H}) = \sum_{H \in \mathcal{H}} |H|$.
- *Edge-diameter sum:* $D(\mathcal{H}) = \sum_{H \in \mathcal{H}} \max \{j - k \mid x_j, x_k \in H\}$.
  That is, diameter of an edge H is the difference $j - k$ where j is the largest and k is the smallest index occurring at the vertices of H.
- *Total edge-distance sum:* $Td(\mathcal{H}) = \sum_{H \in \mathcal{H}} \sum_{x_l \in H} (\max \{j \mid x_j \in H\} - l)$.
  In each edge H the distances between the last vertex and the other ones are summed.
- *Reverse-index sum:* $R(\mathcal{H}) = \sum_{H \in \mathcal{H}} \sum_{x_l \in H} (n + 1 - l)$.
  This means that in the unique UC-order $x_1, x_2, \ldots, x_n$, new descending indices from n to 1 are introduced. So, each vertex x_l has reverse-index $(n+1-l)$ and every edge is represented by the sum of reverse-indices assigned to its vertices.

Let us note that if the vertices of a mixed hypergraph $\mathcal{H}$ are colored in the order $x_1, x_2, \ldots, x_n$, then the reverse sequence $(y_1, \ldots, y_n) = (x_n, \ldots, x_1)$ is termed the *elimination order* of $\mathcal{H}$ (cf. [59]). That is, in $R(\mathcal{H})$ each edge is measured by the sum of the indices of its vertices in the elimination order of $\mathcal{H}$.

If we study all the UUC-graphs belonging to a given color-order, it can be proved, that the UUC-graph $\mathcal{H}^*$ constructed in the previous proof is minimal under each of the five measures.

Proposition 8. *For any given color-order (according to conditions C_0, C_1, C_2) the constructed $\mathcal{H}^*$ is a minimal UUC-graph concerning the measures N, S and D. Moreover, $\mathcal{H}^*$ is the only minimal UUC-graph under Td and R.*

The proof can be found in paper [2].

6 Color-bounded hypergraphs: General results

In the previous chapters we studied mixed hypergraphs where two types of coloring constraints are used. In a proper coloring, the vertices of a $\mathcal{D}$-edge have to get at least two different colors, whilst inside a $\mathcal{C}$-edge C, there can occur at most $|C|-1$ different colors. These two opposite types of coloring conditions make it possible to model many kinds of practical problems.

But there is also a wide range of problems where these two types of constraints are not sufficiently strong for exact modeling. For instance, in many applications there are prescribed at least three or at least four different types (colors) for certain groups. A characteristic example is when we need multiple controls for security reasons. From the other side, we may also have constraints on the maximum number of used colors, for instance the number of allocatable resources is usually bounded in a given field. In fact, this latter type of conditions can also be expressed by using $\mathcal{C}$-edges, but this formulation would increase the number of hyperedges drastically. For a ten-element group E_i, the upper bound five on the number of used colors would mean to consider all the 210 six-element subsets of E_i as $\mathcal{C}$-edges. (Due to the results of Chapter 3, this can be expressed using 'only' about 70 subsets, but the structure would become very complicated.)

Consequently, the introduction of color-bounded hypergraphs is motivated not only from theoretical but also from practical side. In this model we can prescribe bounds s_i and t_i on each edge E_i, which force that E_i receives at least s_i and at most t_i different colors in every proper coloring. It is clear that this model generalizes the concept of mixed hypergraph, and we will see in Chapters 6 and 7 that a much stronger model is obtained. We will see examples in Chapter 9 for applications of this new model in informatics.

6.1 Preliminary results

In Chapter 1 we have already introduced our new hypergraph coloring model and defined the notion of color-bounded hypergraph. In this introductory section of the present chapter we make some simple observations. First, connections of color-bounded hypergraphs with mixed hypergraphs are considered. After that, we determine the condition of colorability, and the lower and upper chromatic number of complete uniform hypergraphs with uniform color-bounds.

6.1.1 Simple reductions

There are particular situations where a color-bounded hypergraph can be reduced to one with smaller edges, or even to a *mixed* hypergraph. Below we mention such cases, collected in one assertion of several parts. Throughout, $\mathcal{C}$-edges and $\mathcal{D}$-edges are meant in the sense of mixed hypergraphs.

Remark 6. *Let $\mathcal{H}$ be a color-bounded hypergraph. If no structural conditions are imposed, then the following reductions can be applied.*

(a) If $s_i = t_i = |E_i|$, then one may replace E_i with all of its 2-element subsets as $\mathcal{D}$-edges, because every feasible coloring of E_i assigns mutually distinct colors to its vertices.

(b) If $s_i = 1$ and $t_i < |E_i| - 1$, then one may replace E_i with all of its $(t_i + 1)$-element subsets as $\mathcal{C}$-edges, because the original E_i is colored in a feasible way if and only if no $t_i + 1$ of its vertices are totally multicolored.

(c) More generally, if $t_i < |E_i| - 1$, then one may modify the bounds (s_i, t_i) of E_i to $(s_i, |E_i|)$ and insert all the $(t_i + 1)$-element subsets of E_i, with bounds $(1, t_i)$.

(d) In parts (b) and (c) it suffices to take all the $(t_i + 1)$-subsets incident with one arbitrarily chosen vertex of E_i, because in any coloring, each vertex of E_i can be completed to a largest polychromatic subset of the edge.

There is a consequence of these reductions that we find worth mentioning separately:

Corollary 6. *If $\mathcal{H}$ does not have any edges such that $|E_i| > 3$ and $s_i > 2$ hold simultaneously, then there exists a* <u>*mixed*</u> *hypergraph on the same vertex set, with precisely the same proper colorings as $\mathcal{H}$.*

Proof If $s_i = 1$, then reduction (b) replaces E_i with $\mathcal{C}$-edges of cardinality $t_i + 1$. The situation is similar if $s_i = 2$, except that E_i itself becomes a $\mathcal{D}$-edge under reduction (c) in this case. Finally, if $s_i \geq 3$, then $|E_i| \leq 3$ by assumption, and so the condition $s_i \leq t_i \leq |E_i|$ implies $s_i = t_i = |E_i| = 3$. Hence, the edge can be replaced with three 2-element $\mathcal{D}$-edges, applying reduction (a). $\square$

It is important to note that the reductions listed in Remark 6 may lead out from a particular class of hypergraphs. For example, as it will be shown in Chapter 7, already the 3-uniform color-bounded hypertrees differ substantially from mixed hypertrees.

Let us note further that the edges with $2 < s_i < |E_i| = t_i$ usually *cannot be replaced* with any combinations of $\mathcal{D}$-edges. This is the reason of the fact (demonstrated in Section 6.3) that assuming a fixed number of vertices, some chromatic spectra are possible for color-bounded hypergraphs but cannot occur in mixed hypergraphs.

6.1.2 Complete uniform hypergraphs

Here we consider the following particular example.

Definition. Let $n \geq r \geq t \geq s \geq 1$ be integers. The *complete uniform color-bounded hypergraph* $\mathcal{K}_n(r;s,t)$ is the 4-tuple $(X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ such that X consists of n vertices, and the edge set $\mathcal{E}$ contains all the r-element subsets of X with associated constant bounds $s_i = s$ and $t_i = t$.

Proposition 9. *The hypergraph $\mathcal{K}_n(r;s,t)$ is colorable if and only if*

(*i*) $t = r$, *or*

(*ii*) $s = 1$, *or*

(*iii*) $n \leq r - 1 + \left\lfloor \frac{r-1}{s-1} \right\rfloor (t - s + 1).$

Proof For $t = r$, the hypergraph is evidently colorable with n colors; and if $s = 1$, then all vertices can get the same color. This shows the sufficiency of the first two cases.

Suppose next that $\mathcal{K}_n(r;s,t)$ is colorable and that $t \neq r$, $s \neq 1$ hold. Assuming a coloring with at least $t+1$ colors, this would yield some edges colored with more than t colors (since $t < r$). This is forbidden, hence there appear at most t color classes.

If there were $s-1$ color classes whose union consists of at least r vertices, they would determine an edge with fewer than s colors. Therefore, even the $s-1$ largest color classes can have at most $r-1$ vertices in total. The remaining color classes are of size at most $\lfloor \frac{r-1}{s-1} \rfloor$ each, and their number is at most $t - (s-1)$. Summing up, if $\mathcal{K}_n(r;s,t)$ is colorable, then

$$n \leq r - 1 + (t - s + 1) \left\lfloor \frac{r-1}{s-1} \right\rfloor$$

necessarily holds.

From now, we assume that $t \neq r$, $s \neq 1$, and $n \leq r - 1 + (t-s+1) \lfloor \frac{r-1}{s-1} \rfloor$. We need to prove that the hypergraph is colorable. Let us write $(r-1)$ in the form $r - 1 = a(s-1) + m$, where a and $m < s - 1$ are nonnegative integers. Applying that $a = \lfloor \frac{r-1}{s-1} \rfloor$ and $m = r - 1 - (s-1) \lfloor \frac{r-1}{s-1} \rfloor$, we get the assumption rewritten in the form $n \leq at + m$. Hence, for $n' = at + m$ vertices there exists a vertex partition $X = X_1 \cup \cdots \cup X_t$ such that X_i has cardinality $a+1$ for $1 \leq i \leq m$, and cardinality a for $m + 1 \leq i \leq t$. Moreover, a partition with t nonempty classes can be obtained for n vertices, too, by removing $n' - n$ vertices in any way so that at most $|X_i| - 1$ are deleted from each X_i.

It remains to observe that this vertex partition colors $\mathcal{K}_n(r;s,t)$ properly. Indeed, the union of the $s-1$ largest classes has at most $a(s-1) + m = r - 1$ elements, therefore every edge contains at least s colors. Moreover, there are just t colors, so that the upper bound is respected, too. This completes the proof. □

Remark 7. *Along the proof, we have also obtained that the upper chromatic number of a colorable color-bounded hypergraph $\mathcal{K}_n(r; s, t)$ is*

- *t, if $t < r$, and*
- *n, if $t = r$.*

More generally, one may consider the complete (p, q)-uniform (s, t)-hypergraphs $\mathcal{K}_n(p, q; s, t)$, where all p-subsets of the underlying n-set have color-bounds (s, p) and all the q-subsets have bounds $(1, t)$. Here we assume $q \geq t \geq s$ and $p \geq s$. The case $q = t$ is already included above, because it means no real upper bound and hence it is equivalent to $p = q = t$. On the other hand, if $q > t$, it is readily seen that every q-element subset contains at most t colors if and only if so does every $(t+1)$-element subset. Consequently, $\overline{\chi}(\mathcal{K}_n(p, q; s, t)) = \overline{\chi}(\mathcal{K}_n(p, t+1; s, t)) \leq t$ holds, and these two hypergraphs are chromatically equivalent. (In particular, both are colorable or both are uncolorable.) For this reason, the value of q is irrelevant and, along the lines of the proof above, we obtain

Proposition 10. *The color-bounded hypergraph $\mathcal{K}_n(p, q; s, t)$ is colorable if and only if*

(i) $t = q$, *or*

(ii) $s = 1$, *or*

(iii) $n \leq p - 1 + \left\lfloor \dfrac{p-1}{s-1} \right\rfloor (t - s + 1).$ □

Putting $s = 2$ and $t = q - 1$, the following simple inequality (first observed in [54]) occurs as a particular case.

Corollary 7. *The complete mixed hypergraph on n vertices, with all p-element subsets as $\mathcal{D}$-edges and all q-element subsets as $\mathcal{C}$-edges is colorable if and only if $n \leq (p-1)(q-1)$.*

Already this rather restricted structure indicates that color-bounded hypergraphs are much more complex than mixed ones: while the latter would admit a one-line proof, the former needs some work.

We have already observed that the $s-1$ largest color classes can have at most $r-1$ vertices. As regards the lower chromatic number, this leads — in the same way as above — to

Proposition 11. *For every $r \geq s \geq 2$ we have*

$$\chi(\mathcal{K}_n(r;s,r)) = s-1+\left\lceil \frac{n-r+1}{\left\lfloor \frac{r-1}{s-1} \right\rfloor} \right\rceil$$

and, if t is not smaller than this lower bound and $q > t$, or $t \geq s$ and $q = t$, then

$$\chi(\mathcal{K}_n(r,q;s,t)) = s-1+\left\lceil \frac{n-r+1}{\left\lfloor \frac{r-1}{s-1} \right\rfloor} \right\rceil. \qquad \square$$

6.2 Chromatic polynomials

Most of the results proved in this section are valid in the general models of pattern and stably bounded hypergraphs, therefore we do not restrict ourselves to color-bounded ones here.

In *pattern hypergraphs*, each *hyperedege* is associated with a collection of proper color partitions on its set of vertices, and a vertex coloring of the entire hypergraph is proper if and only if so is the induced color partition on each edge. This concept was studied by Dvořák *et al.* [20], their main result characterizes those pattern types which admit gaps in the feasible set.

There is a natural way to assign values s_i, t_i to the edges E_i of a pattern hypergraph, too. Namely, for each edge we can take the smallest and largest numbers of nonempty classes, over the feasible color partitions of E_i. Similarly, the parameters χ, $\overline{\chi}$, and the set Φ have their obvious meaning for every colorable pattern hypergraph. Having these definitions at hand, the main results of this section remain valid for this most general model, too.

Here stably bounded hypergraphs can be viewed as particular pattern hypergraphs and the obtained values s'_i and t'_i may be different from the originally given ones, e.g., $s_i = 1$ and $b_i = \frac{|E_i|}{2}$ will mean that $s'_i = 2$.

Let us extend the standard notion of chromatic polynomial $P(\mathcal{H}, \lambda)$ for these widest classes of hypergraphs. By definition, the value of $P(\mathcal{H}, \lambda)$ for a positive integer $\lambda = k$ is the number of proper colorings with at most k colors where we count permutations of colors to be distinct.

For convenience, we shall use some standard notation:

- For $n \geq k > 0$ we denote by $S(n,k)$ the *Stirling number of the second kind*, which is the number of partitions of n elements into precisely k nonempty sets.

- The denotation $[\lambda]_k := \lambda(\lambda-1)\cdots(\lambda-k+1)$ stands for a 'falling power'.
- Stirling numbers and falling powers are connected in the following fundamental equation:

$$\lambda^n = \sum_{k=1}^{n} S(n,k) \cdot [\lambda]_k \tag{2}$$

It is clear that if $\mathcal{H}$ is uncolorable, then its chromatic polynomial is $P(\mathcal{H},\lambda) \equiv 0$, the identically zero function. For this reason, we assume throughout this section that $\mathcal{H}$ is *colorable*. (It does not mean that hypergraphs derived from $\mathcal{H}$, too, will be assumed to be colorable.) The chromatic polynomial will be written in the form

$$P(\mathcal{H},\lambda) = \sum_{k\geq 0} a_k \lambda^k$$

After recalling some known facts from the literature, we shall first observe that this is a legal notation, as the number of colorings with at most λ colors is indeed a polynomial of λ. For an easier formulation of some assertions, we shall use the notation

$$\mathcal{E}^{(r)} = \{E_i \in \mathcal{E} \mid |E_i| = r\}$$

and $m_r = |\mathcal{E}^{(r)}|$, for all $r \geq 2$. In particular, $\mathcal{E}^{(2)}$ is the graph formed by the 2-element edges of $\mathcal{H}$.

Known facts

1. If $s_i = 2$ and $t_i = |E_i|$ for all i (that is, hypergraph coloring in the usual sense), then

   (a) $P(\mathcal{H},\lambda)$ is a polynomial of order $n = |X|$,

   (b) $a_n = 1$ and $a_0 = 0$,

   (c) $\sum_{k\geq 0} a_k = 0$, if there exists at least one edge,

   (d) $a_{n-1} = -m_2$, where $m_2 = |\mathcal{E}^{(2)}|$ (Dohmen [18]).

   (e) $a_{n-2} = \binom{m_2}{2} - m_3 - t_2$, where t_2 is the number of triangles in the graph $\mathcal{E}^{(2)}$, provided that no $E_i \in \mathcal{E}^{(2)}$ is a subset of any $E_j \in \mathcal{E}^{(3)}$. This follows from a general equation of [18], although is not stated there explicitly.

2. Facts (1a)–(1e) can be extended to hypergraphs such that

$$s_i \geq 2 \qquad \text{and} \qquad t_i = |E_i| \qquad \forall\, 1 \leq i \leq m,$$

   with some modifications in (1d) and (1e). In the present case, each edge E_i of $\mathcal{H}$ having $s_i = |E_i|$ is replaced with its 2-element subsets as edges, and $\mathcal{E}^{(r)}$ for $r \geq 2$ is meant to be the collection of r-element edges *after* this operation. Then a_{n-1} is determined as in (1d), and the formula of (1e) for a_{n-2} is proved to be valid under the further condition $s_i \leq |E_i| - 2$ for all $E_i \notin \mathcal{E}^{(2)}$, which also implies that $m_3 = 0$ holds (Drgas-Burchardt and Łazuka [19]).

3. If $\mathcal{H}$ is a mixed hypergraph, then $P(\mathcal{H},\lambda) = \sum_{k=\chi}^{\overline{\chi}} r_k[\lambda]_k$ (Voloshin [58]).

Turning now to color-bounded, and, more generally, stably bounded and pattern hypergraphs, we begin with the following very general observation.

Proposition 12. *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph, and $\mathcal{P}$ a given set of partitions of X, whose members are the allowed color partitions for $\mathcal{E}$. For $k = 1, \dots, |X|$ denote by r_k the number of those partitions in $\mathcal{P}$ which have precisely k nonempty classes. Then the number $P(\mathcal{H},\lambda)$ of allowed colorings of $\mathcal{H}$ with at most λ colors is a polynomial of λ, and it can be written as $P(\mathcal{H},\lambda) = \sum_{k>0} r_k[\lambda]_k$.*

Proof A unique ordering can be assigned to the classes of each partition in $\mathcal{P}$, according to their vertex of smallest subscript in the order $x_1, \dots, x_n$. Enumerating along this ordering, if $P \in \mathcal{P}$ has precisely k classes, then there are exactly $[\lambda]_k$ ways to assign distinct colors to its classes. □

Corollary 8. *For every $\mathcal{H}$, $P(\mathcal{H},\lambda)$ is a polynomial of order $\overline{\chi} = \overline{\chi}(\mathcal{H})$ whose leading coefficient equals $r_{\overline{\chi}}$, that is the number of color partitions with the maximum number of colors.*

Corollary 9. *Two hypergraphs are chromatically equivalent if and only if they have the same chromatic polynomial.*

The following fact has been proved in [19] for the hypergraphs where $t_i = |E_i|$ holds for all edges.

Corollary 10. *Every chromatic polynomial can be written as the sum of chromatic polynomials of <u>graphs</u>, without any negative coefficients.*

Proof It suffices to observe that $[\lambda]_k$ is the chromatic polynomial of the complete graph on k vertices, and that none of the $[\lambda]_k$ can have a negative coefficient in $P(\mathcal{H},\lambda)$. □

Remark 8. *By definition, all integers i with $1 \le i < \chi(\mathcal{H})$ are roots of $P(\mathcal{H},\lambda)$.*

Corollary 11. *If $\mathcal{H}$ is colorable and $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H})$, then the set of roots of $P(\mathcal{H},\lambda)$ is*

$$\{i \in \mathbb{N} \cup \{0\} \mid i \le \chi(\mathcal{H}) - 1\}$$

and each root has multiplicity one; and vice versa, if the roots of $P(\mathcal{H},\lambda)$ are $0, 1, \dots, k-1$ and each of them has multiplicity one, then $\mathcal{H}$ is colorable and $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H})$.

Proof If $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = k > 0$, then $P(\mathcal{H},\lambda) = r_k \cdot \lambda(\lambda-1)\cdots(\lambda-k+1)$ by Proposition 12. Conversely, the given set of roots (without multiple roots) implies that $P(\mathcal{H},\lambda)$ is of the form $r_k \cdot \lambda(\lambda-1)\cdots(\lambda-k+1)$, from which we obtain that $r_i = 0$ for every $1 \le i < k$ and also for $i > k$, so that k is the unique possible number of colors. □

For the explicit computation of $P(\mathcal{H}, \lambda)$, let us introduce the following notation:

- $\mathcal{H} + C_{i,j}$ — the hypergraph obtained from $\mathcal{H}$ by inserting the $\mathcal{C}$-edge $\{x_i, x_j\}$ with $s = t = 1$
- $\mathcal{H} + D_{i,j}$ — the hypergraph obtained from $\mathcal{H}$ by inserting the $\mathcal{D}$-edge $\{x_i, x_j\}$ with $s = t = 2$

In order to compute $P(\mathcal{H}, \lambda)$, the following recursion may be useful.

Proposition 13. *If $\{x_i, x_j\}$ is neither a $\mathcal{C}$-edge nor a $\mathcal{D}$-edge, then*

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} + C_{i,j}, \lambda) + P(\mathcal{H} + D_{i,j}, \lambda).$$

Proof The first and second term on the right-hand side counts the number of those colorings of $\mathcal{H}$ with at most λ colors in which x_i and x_j get the same color or distinct colors, respectively. □

In the same way, one can also observe that

$$\overline{\chi}(\mathcal{H}) = \max\left(\overline{\chi}(\mathcal{H} + C_{i,j}), \overline{\chi}(\mathcal{H} + D_{i,j})\right)$$

for every $\mathcal{H}$. The analogous formula

$$\chi(\mathcal{H}) = \min\left(\chi(\mathcal{H} + C_{i,j}), \chi(\mathcal{H} + D_{i,j})\right)$$

is valid under the slight restriction that both hypergraphs $\mathcal{H}+C_{i,j}$ and $\mathcal{H}+D_{i,j}$ on the right-hand side are colorable; otherwise the uncolorable one has to be omitted and the lower chromatic number is equal to that of the colorable modified hypergraph.

Remark 9. *This recursion leads to an analogue of the 'Splitting–Contraction Algorithm' developed by Voloshin for mixed hypergraphs [57].*

Proposition 14. *If $\max_{1 \le i \le m} s_i \ge 2$, then the sum of the coefficients of $P(\mathcal{H}, \lambda)$ is equal to zero; and if $s_1 = ... = s_m = 1$, then the sum is equal to 1.*

Proof The value $P(\mathcal{H}, 1)$ is equal to the sum of coefficients, and at the same time it counts the number of allowed colorings with just one color. The latter obviously is either 0 or 1. □

Proposition 15. *The constant term in $P(\mathcal{H}, \lambda)$ is equal to zero.*

Proof According to Proposition 12, each term in the formula for $P(\mathcal{H}, \lambda)$ is divisible by λ. □

The following example shows that Facts (1d) and (1e) do not remain valid if some edges have $t_i < |E_i|$.

Example 1. Let $\mathcal{H}$ have the only one edge, $E_1 = X$, with $s_1 = 1$ and $t_1 = n-1$. Then

$$P(\mathcal{H},\lambda) = \lambda^n - \prod_{i=0}^{n-1}(\lambda - i) = \lambda^{n-1}\sum_{i=0}^{n-1} i - \lambda^{n-2}\sum_{i=1}^{n-1}\sum_{j=0}^{i-1} ij + O(\lambda^{n-3})$$

$$= \binom{n}{2}\lambda^{n-1} - \left(\frac{1}{2}\binom{n}{2}^2 - \frac{1}{8}\binom{2n}{3}\right)\lambda^{n-2} + O(\lambda^{n-3})$$

because

$$\sum_{j<i\le n-1} ij = \sum_{i\le n-1} i\binom{i}{2} = \frac{1}{2}\sum_{i\le n-1} i^3 - \frac{1}{2}\sum_{i\le n-1} i^2$$

$$= \frac{1}{8}n^2(n-1)^2 - \frac{1}{12}n(n-1)(2n-1)\,. \qquad \square$$

The following result completely characterizes the chromatic polynomials under the assumption $P(1) = 0$.

Theorem 9. *Let $P(\lambda) = \sum_{k=0}^{\ell} a_k\lambda^k \not\equiv 0$ be a polynomial such that $P(1) = 0$, i.e. $\sum_{k=0}^{\ell} a_k = 0$. The following properties are equivalent.*

1. *$P(\lambda)$ is the chromatic polynomial of a color-bounded hypergraph.*
2. *$P(\lambda)$ is the chromatic polynomial of a mixed hypergraph.*
3. *$P(\lambda)$ is the chromatic polynomial of a stably bounded hypergraph.*
4. *$P(\lambda)$ is the chromatic polynomial of a pattern hypergraph.*
5. *$P(\lambda)$ satisfies all of the following conditions.*
   - *(i) All coefficients a_k of $P(\lambda)$ are integers.*
   - *(ii) The leading coefficient a_ℓ is positive.*
   - *(iii) The constant term a_0 is zero.*
   - *(iv) For every positive integer $j \le \ell$, the inequality*

     $$\sum_{k=j}^{\ell} a_k \cdot S(k,j) \ge 0$$

     is valid.

Proof The condition $P(1) = 0$ means that if some hypergraph $\mathcal{H}$ has $P(\lambda)$ as its chromatic polynomial, then at least one edge $E_i \in \mathcal{H}$ has $s_i \ge 2$. In particular, if $\mathcal{H}$ is such a *mixed* hypergraph, then it contains at least one $\mathcal{D}$-edge. Due to the construction in [32], prescribing the numbers of k-colorings in an arbitrary way for

$k = 2, \ldots, \overline{\chi}$, there exists a mixed hypergraph with this given chromatic spectrum (provided that $r_1 = 0$, which is now the case). Since this already includes all possible variations, the equivalence of properties 1, 2, 3, and 4 follows.

In order to prove the equivalence of property 5 with 1 through 4, let us introduce the notation $r_j = \sum_{k=j}^{\ell} a_k \cdot S(k,j)$ for $j = 1, \ldots, \ell$. On applying Eq. (2), we rewrite

$$P(\lambda) = \sum_{k=1}^{\ell} a_k \lambda^k = \sum_{k=1}^{\ell} a_k \sum_{j=1}^{k} S(k,j) \cdot [\lambda]_j$$

$$= \sum_{j=1}^{\ell} [\lambda]_j \sum_{k=j}^{\ell} a_k \cdot S(k,j) = \sum_{j=1}^{\ell} r_j \cdot [\lambda]_j \qquad (3)$$

It is clear that $r_\ell = a_\ell$. Moreover, all the a_k are integers if and only if all the r_k are integers. Indeed, the *largest* subscript j for which r_j is not an integer would yield that a_j is not an integer either; and vice versa.

Suppose now that $P(\lambda)$ is equal to $P(\mathcal{H}, \lambda)$ for some mixed hypergraph $\mathcal{H}$. By Proposition 12, the meaning of r_j is the number of color partitions into precisely j nonempty vertex classes. Consequently, $r_j \geq 0$ must hold for all j, which implies (iv). Also, the condition $a_0 = 0$ follows by Proposition 15.

Conversely, suppose that the conditions $5(i)$–$5(iv)$ are valid for $P(\lambda)$. Applying Eq. (3) we obtain a sequence $(r_j)_{j>0}$ of nonnegative integers. We observe now that $r_1 = 0$ necessarily holds; this follows from the assumption $P(1) = 0$, because the right-hand side of (3) yields $P(1) = r_1$. Thus, due to [32], there exists a mixed hypergraph $\mathcal{H}$ whose chromatic spectrum is $(r_j)_{j>0}$. This completes the proof of the theorem. □

Corollary 12. *For every color-bounded hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ there exists a mixed hypergraph with the same chromatic polynomial and with the same chromatic spectrum.*

Proof For the case $\max_{1 \leq i \leq m} s_i \geq 2$, this is just the equivalence of properties 1 and 2 above. In the other case, if $s_1 = \ldots = s_m = 1$, we replace each E_i with all its $(t_i + 1)$-element subsets as $\mathcal{C}$-edges, as described in reduction (b) of Remark 6. The color partitions of this 'mixed' $\mathcal{C}$-hypergraph are precisely those of $\mathcal{H}$. □

We shall see later that this property does not remain valid anymore when the hypergraph under consideration is assumed to be uniform. It is not valid either in the class of stably bounded hypergraphs, as will be proved in Chapter 8.

Concerning pattern hypergraphs, 1-colorability does not imply that the chromatic spectrum is gap-free. This makes it possible to characterize the chromatic polynomials completely.

Proposition 16. *The polynomial $P(\lambda) = \sum_{i=0}^{\ell} a_k \lambda^k$ is the chromatic polynomial of some pattern hypergraph if and only if all of the following conditions hold.*

(i) *All coefficients* a_k *of* $P(\lambda)$ *are integers.*

(ii) *The leading coefficient* a_ℓ *is positive.*

(iii) *The constant term* a_0 *is zero.*

(iv) *For every positive integer* $j \leq \ell$*, the inequality*

$$\sum_{k=j}^{\ell} a_k \cdot S(k,j) \geq 0$$

is valid.

(v) $\sum_{i=0}^{\ell} a_k = 0$ *or* 1.

Proof The number of 1-colorings, that is the value $P(1) = \sum_{i=0}^{\ell} a_k$, is either 0 or 1, yielding the necessity of (v). For the case of $P(1) = 0$, the proof of Theorem 9 verifies the assertion for pattern hypergraphs as well. Moreover, the argument given there also implies that the conditions (i)–(iv) are necessary for the case $P(1) = 1$, too.

As regards $P(1) = 1$, the crucial point is that for the 1-colorable pattern hypergraphs (equivalently when $r_1 = 1$), the other entries of the chromatic spectrum can be arbitrarily prescribed nonnegative integers. To prove this, consider the nonnegative integers $r_2, \ldots, r_\ell$ and let a sufficiently large n be chosen such that $S(n,k) \geq r_k$ holds for each $k = 2, \ldots, \ell$. Create a hypergraph $\mathcal{H} = (X, \{X\})$ on n vertices. By the choice of n, we can prescribe for the hyperedge X exactly r_k feasible k-partitions for every $1 \leq k \leq \ell$, what results in a pattern hypergraph $\mathcal{H}$ with chromatic spectrum $(r_1 = 1, r_2, \ldots, r_\ell)$.

From this point the proof can be completed as in the case of Theorem 9. □

6.3 Feasible sets

In this section we study the feasible sets of color-bounded hypergraphs. The main results are the determination of largest gaps in hypergraphs with a given number of vertices, and the characterization of feasible sets of *uniform* color-bounded hypergraphs (Theorem 11).

It was proved in [29] that *any* finite set of integers greater than 1 is the feasible set of a mixed hypergraph, but the smallest number of vertices realizing a given set is not known. For gaps of size k, however, a construction on $2k+4$ vertices was given in [29]. This $2k+4$ is the smallest possible order, what follows from another result of the same paper, although it was not stated explicitly until [6]. On the other hand, for color-bounded hypergraphs the minimum is much smaller, as shown by the following tight result.

Theorem 10. *If a color-bounded hypergraph has a gap of size $k \geq 1$ in its chromatic spectrum, then it has at least $k+5$ vertices. Moreover, this bound is sharp; that is, for every positive integer k there exists a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ with $|X| = k+5$ vertices whose chromatic spectrum has a gap of size k.*

Proof The proof can be found in [4]. On the other hand, in Chapter 8 we will prove Theorem 21 which immediately implies the validity of the present theorem without vicious circles.

Remark 10. *In pattern hypergraphs, the minimum order for a gap of size k is equal to $k+2$. This bound is attained by the hypergraph $\mathcal{H} = (X, \{X\})$ with $|X| = k+2$ and $r_1 = r_{k+2} = 1$, $r_2 = \ldots = r_{k+1} = 0$; i.e., where the vertex set is required to be either monochromatic or completely multicolored.*

As it was shown in Section 6.2, the classes of mixed and color-bounded hypergraphs generate the same set of chromatic polynomials; moreover, the feasible sets are the same in any case. In hypergraphs with restricted structures, however, there appear substantial differences. We consider the following three types, the third one being the main issue of this subsection.

- Hypertrees

  The chromatic spectrum of mixed hypertrees is gap-free and their lower chromatic number is at most two [34]. Hence, their feasible sets determine precisely the intervals of the form $[1, \ldots, k]$ or $[2, \ldots, \ell]$, where $k \geq 1$ and $\ell \geq 2$.

  On the other hand, we shall prove in Chapter 7 that color-bounded hypertrees can have arbitrary large gaps in the chromatic spectrum, and any positive integer can occur as a lower chromatic number. Any set S of integers with $\min S \geq 3$ can be obtained as the feasible set of some color-bounded hypertree; if the lower chromatic number equals 1 or 2, however, then the chromatic spectrum is necessarily gap-free.

- Interval hypergraphs

  Mixed interval hypergraphs have a gap-free chromatic spectrum with lower chromatic number 1 or 2 [29], whilst in the color-bounded case the spectrum still remains gap-free but the lower chromatic number can be any positive integer, according to our Theorem 14.

- r-uniform hypergraphs

  The 2-uniform mixed and color-bounded hypergraphs are practically the same: the $(1,2)$-edges have no effect on coloring, and after their deletion we get a 2-uniform mixed hypergraph (i.e., a 'mixed graph').

  The larger cases, where $r \geq 3$, will be treated in this subsection and essential differences will be demonstrated in comparison with mixed hypergraphs.

We now consider the feasible sets belonging to r-uniform hypergraphs, for different values of r.

For the next observations the classes of possible feasible sets will be denoted by $\mathcal{F}_m$ and $\mathcal{F}_c$ regarding mixed and color-bounded hypergraphs, respectively. When we refer only to feasible sets of r-uniform hypergraphs, upper indices will be used: $\mathcal{F}_m^r$ and ${\mathcal{F}_c}^r$. Since mixed and color-bounded hypergraphs generate the same chromatic polynomials, the sets $\mathcal{F}_m$ and $\mathcal{F}_c$ are equal.

For r-uniform *mixed* hypergraphs, the feasible sets have been characterized in Chapter 2. This implies that the 3-uniform mixed hypergraphs generate all possible feasible sets from $\mathcal{F}_m$, except the set $\{1\}$. Increasing the value of r, for all integers $3 \leq r_1 < r_2$ the inclusion $\mathcal{F}_m^{r_1} \supsetneqq \mathcal{F}_m^{r_2}$ holds. Thus, the classes $\mathcal{F}_m^r$ of possible feasible sets determine a strictly decreasing infinite set-sequence: $\mathcal{F}_m^3 \supsetneqq \mathcal{F}_m^4 \supsetneqq \dots \supsetneqq \mathcal{F}_m^r \supsetneqq \dots$. For every feasible set $S \in \mathcal{F}_m$, there are only finitely many values of r such that an r-uniform mixed hypergraph can have S as its feasible set, since in this case $r \leq 1 + \max S$ necessarily holds. Consequently, there is no feasible set belonging to every element of the above nested sequence.

Contrary to this, we are going to prove that in the case of color-bounded r-uniform hypergraphs the classes ${\mathcal{F}_c}^r$ of possible feasible sets, for all $r \geq 3$, are the same.

Proposition 17. *For every color-bounded hypergraph $\mathcal{H}_1$ having edges only of sizes not larger than r, there exists a chromatically equivalent r-uniform color-bounded hypergraph $\mathcal{H}_2$; that is, $P(\mathcal{H}_1, \lambda) = P(\mathcal{H}_2, \lambda)$.*

Proof Any given hypergraph $\mathcal{H}_1 = (X_1, \mathcal{E}_1, \boldsymbol{s}_1, \boldsymbol{t}_1)$ with edges not larger than r can be extended to $\mathcal{H}_2$ in the following way. For each vertex $x_i \in X_1$, we take additional $r-1$ copies, and join them with x_i in an r-element $(1,1)$-edge to ensure that in each coloring they get the same color as x_i. Then every edge E_j of $\mathcal{H}_1$ can be extended to an r-element edge of $\mathcal{H}_2$ by adjoining $r - |E_j|$ 'copy vertices' of some $x_i \in E_j$ to it, whilst the color-bounds remain unchanged. Clearly, the feasible colorings of $\mathcal{H}_1$ and $\mathcal{H}_2$ are in one-to-one correspondence, thus the two hypergraphs have the same chromatic polynomial. □

Proposition 18. *For each integer $r \geq 3$, the r-uniform color-bounded hypergraphs generate all possible feasible sets from $\mathcal{F}_c$.*

Proof For $r = 3$, already the mixed hypergraphs generate all the feasible sets from $\mathcal{F}_m$ except the set $\{1\}$. Obviously, the 3-uniform color-bounded hypergraphs determine all these feasible sets and also the set $\{1\}$. A trivial example for the latter has three vertices joined by a 3-element $(1,1)$-edge. Due to Corollary 12, $\mathcal{F}_m = \mathcal{F}_c$ holds, hence we obtain that $\mathcal{F}_c = {\mathcal{F}_c}^3$. Applying Proposition 17, we obtain for any $r > 3$ that every 3-uniform color-bounded hypergraph has some r-uniform chromatic equivalent, therefore ${\mathcal{F}_c}^r \supseteq {\mathcal{F}_c}^3$. But ${\mathcal{F}_c}^3$ contains all the possible feasible sets of color-bounded hypergraphs, thus for every $r \geq 3$ the equality ${\mathcal{F}_c}^r = \mathcal{F}_c$ holds. □

The class of feasible sets occurring for mixed hypergraphs was characterized in the paper [29]. Combining that result and the above proposition we obtain:

Theorem 11. *For every integer $r \geq 3$, a set S of positive integers is the feasible set of an r-uniform color-bounded hypergraph if and only if*

(i) $\min S \geq 2$, *or*

(ii) $\min S = 1$ *and* $S = \{1, \ldots, k\}$ *for some natural number* $k \geq 1$. □

Comparing the possible feasible sets of r-uniform mixed and color-bounded hypergraphs, we can conclude that for $r = 2$ they are the same (classical graphs), and for $r = 3$ there is only one feasible set — namely, $\{1\}$ — appearing in the color-bounded case and not belonging to any mixed hypergraphs. But increasing the value of r the difference becomes more and more substantial.

Now, we take some observations concerning the chromatic spectra of (r-uniform) color-bounded hypergraphs. It was proven for mixed hypergraphs in [32] that any vector $(r_1, r_2, \ldots, r_k)$ with $r_1 = 0$ and $r_2, \ldots, r_k \in \mathbb{N} \cup \{0\}$ can be obtained as the chromatic spectrum of some mixed hypergraph. In the construction of the proof there occur only $\mathcal{C}$-edges of size 3 and $\mathcal{D}$-edges of size 2. This mixed hypergraph can be considered color-bounded as well, and since it has edges of size not larger than 3, we can apply Proposition 17 to get, for each $r \geq 3$, an r-uniform color-bounded hypergraph with the above chromatic spectrum. As a consequence, we obtain:

Proposition 19. *For every finite sequence $r_2, r_3, \ldots, r_k$ of nonnegative integers and for every $r \geq 3$ there exists some r-uniform color-bounded hypergraph whose chromatic spectrum is $(r_1 = 0, r_2, \ldots, r_k)$.* □

This means that under the assumption $P(1) = 0$, the characterization of chromatic polynomials in Theorem 9 is valid for r-uniform color-bounded hypergraphs, too.

As it was proven in Section 6.2, the possible chromatic polynomials — and also the chromatic spectra — are the same in the case of mixed and color-bounded hypergraphs. Considering a fixed integer $r \geq 3$, however, by our Theorems 1 and 11, there exist feasible sets and hence chromatic spectra, too, occurring for r-uniform color-bounded, but not occurring for r-uniform mixed hypergraphs. We are going to point out that, even assuming a fixed common feasible set and r-uniform hypergraphs, the corresponding spectra can be different.

Let $r = 4$ and the feasible set $\{1, 2, 3\}$ be considered.

For 4-uniform mixed hypergraphs, this means that there occur only $\mathcal{C}$-edges of size 4, hence any partition of the vertex set into 1, 2, or 3 color classes induces a feasible coloring. Therefore $r_1 = 1$, $r_2 = S(n, 2)$, and $r_3 = S(n, 3)$ hold, where n denotes the number of vertices. In particular, $r_1 = 1$ and $r_2 = 15$ together imply that $n = 5$ and $r_3 = 25$.

On the other hand, in 4-uniform color-bounded hypergraphs the 1-colorability means that the lower bound s_i equals 1 for every edge E_i, and after the contraction of $(1, 1)$-edges, any 2-partition of the n vertices yields a proper coloring. Thus $r_1 = 1$ and $r_2 = S(n, 2)$. But if there exists an edge with bounds $(1, 2)$, then not all of the 3-partitions are feasible, hence $0 < r_3 < S(n, 3)$ can hold. Analysis shows that if $r_1 = 1$ and $r_2 = 15$, then the first three entries in the chromatic spectrum belonging to the feasible set $\{1, 2, 3\}$ form one of the following triples:

- $(1, 15, 25)$ — five vertices and four or five 4-element edges, all of them with color-bounds $(1, 3)$;
- $(1, 15, 7)$ — five vertices with one $(1, 2)$-edge of size four;
- $(1, 15, 1)$ — five vertices with two different $(1, 2)$-edges of size four.

As it has been observed, only the first one can belong to 4-uniform mixed hypergraphs.

6.4 Uniquely colorable hypergraphs

A hypergraph $\mathcal{H}$ is called *uniquely colorable* if $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = k$ for some $k \in \mathbb{N}$, and $r_k = 1$. General properties of uniquely colorable *mixed* hypergraphs have been studied in [56] and [13]. Moreover, uniquely $(n-1)$-colorable and uniquely $(n-2)$-colorable mixed hypergraphs are characterized by Niculitsa and Voss in [45]. Though it is co-NP-complete to decide whether a mixed hypergraph (given together with one of its colorings) is uniquely colorable [56], it may be the case that uniquely $(n-c)$-colorable mixed hypergraphs admit a relatively simple structural description for any constant c. In sharp contrast to this, for color-bounded hypergraphs we prove:

Theorem 12. *It is co-NP-complete to decide whether a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ on n vertices is uniquely $(n-1)$-colorable.*

Proof We are going to apply the following result of Phelps and Rödl [47] from the algorithmic theory of balanced incomplete block designs:

> It is NP-complete to decide whether a Steiner Triple System[9], whose blocks are viewed as the $\mathcal{D}$-edges of a 3-uniform (mixed, or 'usual') hypergraph, is colorable with 14 colors.

[9]A Steiner triple system (STS) of order v is a v-element set X together with a set $\mathcal{B}$ of 3-element subsets of X (called blocks) with the property that each 2-element subset of X is contained in exactly one block. With the notation of the footnote on page 8, it is $S(2, 3, v)$, but most standard notation is $STS(v)$.

Let now $X = \{x_1, \ldots, x_{n-2}\}$ be the vertex set of an 'input' Steiner Triple System $\mathcal{S} = STS(n-2) = (X, \mathcal{B})$ of order $n-2$, whose k-colorability (as a $\mathcal{D}$-hypergraph) should be decided for a given integer k. Due to the result quoted above, in our case $k = 14$ will be a suitable choice.

We construct a color-bounded hypergraph $\mathcal{H}$ on the vertex set $X \cup \{z_1, z_2\}$ — i.e., with two new vertices z_1, z_2 — in which the hyperedges and color-bounds are defined as follows:

- $B' = B \cup \{z_1, z_2\}$ with $\boldsymbol{s}(B') = 4$ and $\boldsymbol{t}(B') = 5$, for all blocks $B \in \mathcal{B}$;
- $W' = W \cup \{z_1, z_2\}$ with $\boldsymbol{s}(W') = 1$ and $\boldsymbol{t}(W') = k+2$, for all $(k+1)$-tuples $W \in \binom{X}{k+1}$;
- $E_{i,j} = \{x_i, z_j\}$ with $\boldsymbol{s}(E_{i,j}) = \boldsymbol{t}(E_{i,j}) = 2$, for all $1 \le i \le n-2$ and $j = 1, 2$.

We analyze the colorings φ of $\mathcal{H}$ by considering the following two cases.

Case 1: $\varphi(z_1) = \varphi(z_2)$

Due to the presence of the $\mathcal{D}$-edges $E_{i,j}$, the color-bound functions $\boldsymbol{s}, \boldsymbol{t}$ reduce to the conditions $|\varphi(W)| \le k+1$ for all W and $|\varphi(B)| = 3$ for all B. We may disregard the former, as it does not yield any real restriction. On the other hand, since $\mathcal{S}$ is a Steiner Triple System, the blocks $B \in \mathcal{B}$ cover all vertex pairs, and hence the latter equation means that any two vertices in X must get distinct colors. Thus, X is $(n-2)$-colored and $\mathcal{H}$ is $(n-1)$-colored.

This type of coloring is unique and it exists for any $STS(n-2)$ input $\mathcal{S}$; and it obviously colors the constructed $\mathcal{H}$ with a proper $(n-1)$-coloring.

Case 2: $\varphi(z_1) \ne \varphi(z_2)$

Then the conditions reduce to $|\varphi(W)| \le k$ for all W and $|\varphi(B)| \ge 2$ for all B. Hence, such a coloring exists if and only if the input Steiner system $\mathcal{S}$ admits a coloring with at most k colors. By the theorem quoted above, this is NP-complete to decide.

Given any input $\mathcal{S}$, the color-bounded hypergraph $\mathcal{H}$ together with its $(n-1)$-coloring described in Case 1 can be constructed in polynomial time for any constant k. This $\mathcal{H}$ *is not* uniquely $(n-1)$-colorable if and only if $\mathcal{S}$ *is* k-colorable.

Finally, an n-element set has precisely $\binom{n}{2}$ partitions into $n-1$ nonempty sets, and it can be checked efficiently for each of those partitions whether it is a feasible coloring of $\mathcal{H}$. Moreover, the problem of finding another coloring of $\mathcal{H}$ has its obvious membership in NP. This completes the proof of the theorem. □

6.5 Regular hypergraphs and color-bounded edge colorings of graphs

In the cases of classical and mixed hypergraphs, restricting the vertex degrees to at most 2 or prescribing that any two edges share at most one vertex — that is, *linear* hypergraphs — sometimes makes problems algorithmically easier to handle. For example, there can be given a well-characterized set of obstructions against the colorability of mixed hypergraphs with maximum degree two (Tuza and Voloshin [55]). Efficient algorithms on this class have also been presented (Král', Kratochvíl and Voss [36]).

In contrast to this, we are going to prove that every color-bounded hypergraph can be transformed to a chromatically equivalent 2-regular hypergraph. As a consequence, restricting the vertex degrees to at most 2, the time complexity of colorability problems does not change substantially. For comparison, let us mention that in [36] the mixed hypergraphs were shown to admit chromatically equivalent representations with mixed hypergraphs of maximum degree 3. It follows from known results on complexity that degree 3 cannot be reduced to degree 2, hence color-bounded hypergraphs yield a stronger model in this respect, too.

Proposition 20. *Every color-bounded hypergraph is chromatically equivalent to a 2-regular color-bounded linear hypergraph.*

Proof Consider a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$. If there are vertices with degree 0 or 1, we can create some new edges containing them, with non-restrictive bounds $s = 1$ and $t = |E|$. Thus, we can assume that each vertex of $\mathcal{H}$ has degree at least 2.

To construct a 2-regular hypergraph $\mathcal{H}^+$, for each vertex x_i of $\mathcal{H}$ we create $d(x_i)$ copies and let them form a $(1, 1)$-edge. The edges of $\mathcal{H}$ can be transformed to edges of $\mathcal{H}^+$ in such a way that every vertex is replaced with one of its copies, and every 'copy vertex' occurs in exactly one edge of this type. The color-bounds remain unchanged, and the vertices of $\mathcal{H}$ do not belong to $\mathcal{H}^+$.

Obviously, the copies of a vertex x_i have the same color in every feasible coloring of $\mathcal{H}^+$, so they can be contracted and a feasible coloring of $\mathcal{H}$ is obtained; and vice versa. Thereby, $\mathcal{H}$ and $\mathcal{H}^+$ are chromatically equivalent. Moreover, $\mathcal{H}^+$ is a 2-regular hypergraph, where any two edges either are disjoint or have intersection of size 1, what completes the proof. □

It is important to note that this transformation generally does not preserve the special structural properties (e.g., hypertree, circular hypergraph). On the other hand, some properties can be ensured; for example, the 3-uniformity can be preserved by slightly modifying the construction.

According to the above proposition, it is enough to consider the 2-regular linear color-bounded hypergraphs regarding the general coloring properties. Let us observe that the dual of a 2-regular and linear hypergraph $\mathcal{H}$ is a *simple graph.* Coloring the vertices of $\mathcal{H}$ according to the color-bounds corresponds to an edge-coloring of the dual graph, where each vertex has the same color-bounds (s_i, t_i), as the corresponding edge in $\mathcal{H}$.

Definition. Consider a graph $G = (V, E)$ where each vertex x_i is associated with integer color-bounds: $1 \leq s_i \leq t_i \leq d(x_i)$. A *color-bounded edge-coloring* of G is a mapping from E to $\mathbb{N}$, such that for every vertex x_i, the incident edges are colored with at least s_i and at most t_i distinct colors. In this model it is convenient to assume that G has *no isolated vertices.*

Theorem 13. *Regarding the color-bounded edge-coloring of graphs:*

(i) *A set S of positive integers can be obtained as feasible set if and only if* $\min S \geq 2$ *or* $S = \{1, \ldots, k\}$ *for some* $k \geq 1$.

(ii) *The class of possible chromatic polynomials corresponds to the class of chromatic polynomials occurring for vertex colorings of color-bounded hypergraphs.*

(iii) *These properties remain valid in the restricted class of bipartite graphs, too.*

Proof According to Proposition 20, for every color-bounded hypergraph $\mathcal{H}$, there exists a chromatically equivalent 2-regular, linear color-bounded hypergraph $\mathcal{H}^+$. The dual of $\mathcal{H}^+$ is a simple graph G. The vertex colorings of $\mathcal{H}^+$ are in one-to-one correspondence with the color-bounded edge-colorings of G, provided that each vertex of the latter has the same color-bounds as the corresponding edge in $\mathcal{H}^+$.

Conversely, every graph without isolated vertices has a dual hypergraph, and if there are assigned corresponding color-bounds, the feasible edge-colorings of the former and vertex colorings of the latter determine the same chromatic polynomial. This proves the statement (ii).

Due to (ii), the possible chromatic spectra — and hence the feasible sets, too — are the same for the two structure classes. Taking into consideration the characterization of possible feasible sets of color-bounded hypergraphs, the statement (i) follows.

To prove (iii), it suffices to observe that the dual graph of the constructed hypergraph $\mathcal{H}^+$ is bipartite: the two vertex classes correspond to the 'copy-edges' and the original edges of $\mathcal{H}$, respectively. □

Remark 11. *The validity of Theorem 13 can be extended to color-bounded edge-colorings of* multigraphs*, too. There can appear no additional feasible sets and chromatic polynomials, since also the dual of a multigraph is a hypergraph (though not necessarily linear).*

7 Color-bounded hypertrees and circular hypergraphs

Tree graphs allow us to design much more efficient algorithms than those with unrestricted structures. This is partly true also for mixed hypertrees. Hence, it is an important issue to study the role of hypertrees in the class of color-bounded hypergraphs.

On the one hand, we obtain quite a surprising result: the decision problem of colorability is NP-complete already on 3-uniform color-bounded hypertrees. Thus, we encounter difficulties which did not appear in previous cases. But on the other hand, we point out that nearly all color-bounded hypergraphs can be represented by some hypertree concerning the colorability properties. This also means that color-bounded hypertrees can play a central role in applications, too.

We also consider hypertrees with more restricted structure and identify some subclasses (e.g., interval hypergraphs, RDP-hypertrees) for which the feasible set always is gap-free, and they admit polynomial-time 'recoloring' algorithms.

The Recoloring Lemma, proved here, is an essential tool throughout this chapter. This offers a possibility for designing polynomial-time algorithms that result in new colorings (preferably, using fewer colors) from a known one. It is surprising and very useful that these algorithms can output a new coloring without having explicit knowledge about the hyperedges. The input contains only a proper coloring of the vertices and the largest value of the lower color-bounds prescribed for the hyperedges. Beside the possible practical importance, it turns out to be very useful also theoretically. We apply this tool to determine the possible feasible sets and the lower chromatic number for hypergraphs of various structure classes.

Throughout this chapter, it will be convenient to apply the following simple notation for paths. If the path from vertex a to vertex b is uniquely determined in a graph, the vertex set of this path will be denoted by $[a,b]$. The 'open' and 'half-open' parts of this path are obtained from $[a,b]$ by the omission of both or one of its endpoints:

$$]a,b[\; := [a,b] \setminus \{a,b\}, \quad]a,b] \; := [a,b] \setminus \{a\}, \quad [a,b[\; := [a,b] \setminus \{b\}.$$

7.1 The Recoloring Lemma

This section is devoted to the Recoloring Lemma that will play a crucial role in several algorithmic proofs regarding lower chromatic number and in proving that the chromatic spectrum of certain types of hypergraphs is gap-free.

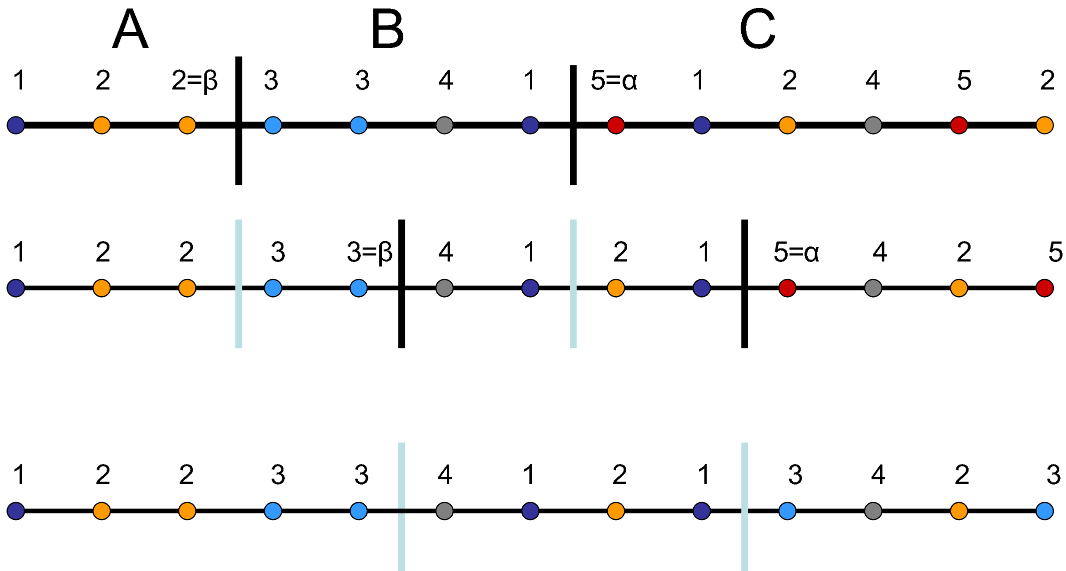

Figure 6: Applying the Recoloring Lemma for an interval hypergraph twice. If we assume that $\max s_i \leq 4$, new colorings can be obtained from a given one without knowing the edges explicitly.

Recoloring Lemma *Let a color-bounded hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ and a proper coloring φ of $\mathcal{H}$ be given. Consider two colors $\alpha, \beta \in \varphi(X)$, a partition of the vertex set X into three parts (A, B, C), and the following set of conditions:*

(1) $\alpha \notin \varphi(B)$ *and* $\beta \notin \varphi(B)$.

(2) *For every hyperedge E_i intersecting both A and C:*

- *(a)* $\alpha \in \varphi(E_i \cap C)$;
- *(b) If* $\alpha \in \varphi(E_i \cap A)$, *then* $\beta \in \varphi(E_i)$;
- *(c)* $|\varphi(E_i \cap B)| \geq s_i - 1$.

If the conditions (1) and (2) hold, then a proper coloring φ' is obtained from φ by transposing colors α and β on the vertex set C.

Proof We prove that the recoloring φ' is proper for every hyperedge of $\mathcal{H}$, whenever the above conditions are met.

- The coloring of hyperedges contained wholly in $A \cup B$ is unchanged, therefore φ' keeps them properly colored.
- By the condition (1), the set B did not contain any vertex colored with α or β. Thus, one may view the recoloring as just switching the denotations of colors

α and β in the entire set $B \cup C$. Therefore, each hyperedge contained in $B \cup C$ has got the same number of colors by φ' as it had originally by φ.

- If a hyperedge E_i intersects both A and C, then, by the condition $2(a)$, color α occurs in $\varphi(E_i)$ and β occurs in $\varphi'(E_i)$. Hence, $|\,|\varphi(E_i)| - |\,\varphi'(E_i)|\,| \leq 1$, and the only possibility for $|\,\varphi'(E_i)| > |\,\varphi(E_i)|$ would be that $\beta \notin \varphi(E_i)$ whilst $\alpha \in \varphi'(E_i)$. By condition $2(b)$, this cannot happen with $\alpha \in \varphi(E_i \cap A)$. Consequently, if there occurs $\alpha \in \varphi'(E_i)$, it has to originate from a vertex colored with β by φ and contained in $E_i \cap C$. That is, if $\varphi'(E_i)$ contains both α and β, then already $\varphi(E_i)$ contained both of them and $|\,\varphi(E_i)| = |\,\varphi'(E_i)|$ holds. Hence, the only possible change in the number of colors occurs when $|\,\varphi'(E_i)| = |\,\varphi(E_i)| - 1$. In this case $\varphi(E_i)$ contained α, β, and at least $s_i - 1$ additional colors from B (due to $2(c)$); hence $|\,\varphi(E_i)| \geq s_i + 1$, whilst $\varphi'(E_i)$ omits the color α and therefore $|\,\varphi'(E_i)| \geq s_i$. Combining these facts, we obtain that $s_i \leq |\,\varphi'(E_i)| \leq |\,\varphi(E_i)| \leq t_i$, consequently E_i is properly colored by φ'.

The above cases cover all possibilities concerning the positions of hyperedges, implying that φ' is a proper coloring of $\mathcal{H}$ indeed. □

We note that some classes of hypergraphs admit a vertex partition (A, B, C) even with the additional property $\alpha \notin \varphi(E_i \cap A)$, under which the condition $2(b)$ automatically holds. Moreover, in our current applications, condition $2(c)$ is used in the weaker form $|\varphi(E_i \cap B)| \geq s - 1$. Nevertheless, we prefer the stronger version to state and prove, with the intention to keep it more general and to allow further potential applications.

7.2 Interval hypergraphs

Considering color-bounded hypergraphs with restricted structures, first we deal with interval hypergraphs and point out some special coloring properties. Throughout this section we assume a host *path* graph with vertex order $x_1, x_2, \ldots, x_n$. Since there exists a unique path from x_i to x_j for every $1 \leq i \leq j \leq n$, we shall write $[x_i, x_j]$ for the 'closed interval' and, analogously, the notation for the 'open' and 'half-open' intervals will be used, too, as introduced at the beginning of this chapter.

It was proved in [29] that no gaps can occur in the chromatic spectrum of any *mixed* interval hypergraph. Our first theorem generalizes this result.

Theorem 14. *Every colorable interval hypergraph $\mathcal{I} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ has a gap-free chromatic spectrum, and its lower chromatic number is equal to $s = \max_{E_i \in \mathcal{E}} s_i$.*

Proof Trivially, if $\mathcal{I}$ is colorable, then its lower chromatic number is at least s. Therefore, to prove both statements of the theorem, it is enough to show that any

k-coloring of $\mathcal{I}$ can be transformed to a proper $(k-1)$-coloring, whenever $k > s$ holds. (This recoloring process is illustrated by an example on Figure 6.)

Consider a k-coloring φ of $\mathcal{I}$ $(k > s)$ and determine the following two vertices. Let a be the vertex for which $[x_1, a]$ is the inclusion-wise minimal starting interval containing all the k colors, that is $|\,\varphi[x_1, a]| = k$ and $\varphi(a) \notin \varphi[x_1, a[$. Then counting the different colors backwards from a in the ordering, choose the vertex b for which $|\,\varphi[b, a]| = s + 1$ and $|\,\varphi]b, a[| = s - 1$.

Next, we show that the conditions of the Recoloring Lemma are fulfilled by

$$A = [x_1, b], \quad B =]b, a[, \quad C = [a, x_n], \quad \alpha = \varphi(a) \quad \text{and} \quad \beta = \varphi(b).$$

- Since $\alpha \notin \varphi[x_1, a[$ and $\beta \notin \varphi\,]b, a[$, the conditions (1) and 2(b) are satisfied.
- If a hyperedge E_i meets both A and C, then it surely contains the interval $[b, a]$. Hence, for this edge $|\,\varphi(E_i \cap B)| = s - 1$ and $\alpha \in \varphi(E_i \cap C)$ hold, complying with 2(c) and 2(a).

Consequently, the Recoloring Lemma can be applied, assuring that the transposition of colors α and β on the interval C yields a proper coloring φ' of $\mathcal{I}$. After this recoloring the interval $[x_1, a]$ has only $k - 1$ colors, and either a $(k-1)$-coloring is obtained, or we still have a k-coloring for which a similar recoloring can be applied again. In the latter case, however, the cardinality of the set C is smaller than it was in the previous recoloring, hence the repeated application of this procedure yields a $(k-1)$-coloring of $\mathcal{I}$ in a finite number of steps.

Starting with $k = \bar{\chi}(\mathcal{I})$, the number of colors can be decreased one by one, thus a coloring is generated for all $k \in \{s, s+1, \ldots, \bar{\chi}(\mathcal{I})\}$. This completes the proof of the theorem. □

Remark 12.

1. *Having a k-coloring ($k > s$) as input of the above algorithm, the number of vertices in $[x_1, a[$ increases by at least one in each recoloring, until the selected color gets eliminated. Therefore it needs at most $O(n)$ phases of recoloring to generate a $(k-1)$-coloring. Moreover, based on the previous proof, one phase can be implemented in $O(n)$ time. Hence, it takes $O(n^2)$ steps to get a $(k-1)$-coloring.*

2. *If our goal is to obtain a χ-coloring from a given k-coloring ($k > \chi = s$), the above algorithm can be slightly modified. In this case in each recoloring phase let a be the vertex for which $[x_1, a]$ is the inclusion-wise minimal starting interval containing $s + 1$ different colors. In this way, we can directly obtain a χ-coloring in $O(n^2)$ time.*

Evidently, every interval $[s,t]$ of positive integers is a feasible set of some color-bounded interval hypergraph. (E.g., consider a hypergraph with only one (s,t)-edge containing all the at least t vertices.) Combining this with Theorem 14, the following characterization is obtained:

Corollary 13. *For a set S of positive integers there exists a color-bounded interval hypergraph $\mathcal{I}$ whose feasible set is $\Phi(\mathcal{I}) = S$ if and only if S is an interval of integers.*

There is an efficient algorithm for *mixed* interval hypergraphs to test their colorability, and also to compute their upper chromatic number [16]. But when the colorings of *color-bounded* interval hypergraphs are considered in general, this seems to be a more difficult problem. Nevertheless, there are some particular types of interval hypergraphs for which we can design polynomial-time algorithms to decide whether they are colorable.

Proposition 21. *If $\mathcal{I}$ is an interval hypergraph satisfying the further condition*

$$s = \max_{E_i \in \mathcal{E}} s_i \le \min_{E_i \in \mathcal{E}} t_i = t$$

then $\mathcal{I}$ is colorable and $\bar{\chi}(\mathcal{I}) \ge t$.

Proof For any $s \le k \le t$, let the vertex x_i get color $i \pmod k$ for $i = 1, 2, \ldots, n$. This periodical coloring is proper for $\mathcal{I}$, because each E_i gets precisely k colors, due to the assumptions $s_i \le s \le k \le t \le t_i \le |E_i|$. □

Let us note that *mixed* interval hypergraphs do belong to this class after the contraction of $\mathcal{C}$-edges of size 2, provided that no $\mathcal{D}$-edge of size 1 arises (that is, the original hypergraph does not contain an obvious obstruction against colorability).

Proposition 22. *The colorability of interval hypergraphs without edges of more than three vertices can be decided in linear time.*

Proof First, we contract each $(1,1)$-edge to one vertex, and then we check whether every edge with $s_i = j$ $(j = 2, 3)$ contains at least j vertices. If this trivial necessary condition holds, then the contracted (and hence also the original) hypergraph is colorable. For example, one can get a proper coloring by the following procedure:

Let $\varphi(x_1) = 1$ and $\varphi(x_2) = 2$; and for $i = 3, \ldots, n$ let $\varphi(x_i) = \varphi(x_{i-2})$ unless x_i is the last vertex of a $(3,3)$-edge. In the latter case, let x_i get a third color, which is different from both previous ones $\varphi(x_{i-1})$ and $\varphi(x_{i-2})$. Then every non-$(3,3)$-edge of the contracted hypergraph gets precisely two colors, hence the coloring is proper. □

Here we only mention, without proof, that there is a greedy coloring algorithm also for another type of color-bounded interval hypergraphs. The constraint is that any two hyperedges of $\mathcal{I}$ should be disjoint or one of them should contain the other. It can be decided in polynomial time whether such a hypergraph is colorable, and if it is, the algorithm computes the upper chromatic number $\overline{\chi}(\mathcal{I})$, and produces a $\overline{\chi}(\mathcal{I})$-coloring.

7.3 Hypergraphs of directed paths

In this section we consider two intermediate classes between unrestricted hypertrees and the interval hypergraphs studied in the previous section.

Rooted Directed Path hypertrees (RDPs). Assume that a root vertex r is fixed in the host tree T, and that each edge of T is oriented in the direction away from the root (i.e., from parent to child). The hypertree $\mathcal{T}$ is termed a *Rooted Directed Path hypergraph* (hypertree) if each hyperedge induces a *directed path* in the rooted host tree. We shall use the shorthand RDP to refer to such structures. It will be assumed throughout that T and $\mathcal{T}$ have the same vertex set, namely $X = \{x_1, \ldots, x_n\}$. An equivalent definition without orienting the edges would be to assume that the vertices within each hyperedge E_i of the hypertree have mutually distinct distances from the root.

Directed Path hypertrees (DPs). A less restricted type of hypertrees is obtained when the host graph is a tree oriented in an arbitrary way, and each hyperedge corresponds to a directed path on the host tree. These hypertrees are termed *Directed Path hypergraphs* (hypertrees), sometimes abbreviated as DP.

Though RDPs and DPs may look fairly similar for the first sight, it will turn out that the former share several special features with interval hypergraphs, while the latter behave quite differently, e.g. regarding gaps and lower chromatic number.

Let us begin with the more restricted subclass of RDPs, which admits results analogous to interval hypergraphs. Having the orientation away from the root fixed, $T(a)$ will denote the subtree rooted at vertex $a \in X$, that is the subtree induced by the set of vertices reachable from a along directed paths.

The interval hypergraphs studied in the previous section are special RDP hypertrees with a path as their host tree, with root $r = x_1$, and with the natural orientation $x_1 \to \ldots \to x_n$. As it was stated in Theorem 14, their chromatic spectrum is gap-free and their lower chromatic number is equal to $\max s_i$.

As shown next, concerning feasible sets and lower chromatic number, RDPs have the same properties.

Theorem 15. *Every colorable RDP has a gap-free chromatic spectrum and its lower chromatic number is equal to $s = \max_{E_i \in \mathcal{E}} s_i$. Moreover, each interval of positive integers can be obtained as a feasible set of some RDP.*

Proof The second part of the theorem is an immediate consequence of Corollary 13. For the first part, let $\mathcal{T} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ be an RDP hypertree. To prove the equation $\chi(\mathcal{T}) = s$ and that there are no gaps in the chromatic spectrum of $\mathcal{T}$, we apply an algorithm based on the Recoloring Lemma.

Consider a k-coloring φ of $\mathcal{T}$, where $k > s$ and the rooted directed host tree is T with root x_1. First, fix a color α from $\varphi(\mathcal{T})$, which is different from the color of the root.

(1) Determine a vertex a of the color class α having smallest distance from the root in T. So, the path $[x_1, a[$ is devoid of color α.

(2) If there exists a color γ not used on the path $[x_1, a]$, then let the colors α and γ be switched on the subtree $T(a)$. If a hyperedge E_i intersects the recolored subtree $T(a)$, it can have 'outside' vertices only from the path $[x_1, a[$, which is devoid of both colors α and γ. Therefore, the colors α and γ were everywhere switched in E_i and the number of colors is unchanged. Hence, the obtained coloring φ' is evidently proper for $\mathcal{T}$ and we can continue with step (4) below.

(3) In the other case when all the k colors ($k \geq s+1$) are used on the path $[x_1, a]$, determine the vertex b from this path, complying with $|\,\varphi[b, a]| = s + 1$ and $|\,\varphi\,]b, a[| = s - 1$.

   To apply the Recoloring Lemma, consider the following sets A, B, C and colors α, β:

$$C = V(T(a)), \quad B =]b, a[, \quad A = X \setminus (B \cup C), \quad \alpha = \varphi(a), \quad \beta = \varphi(b).$$

   - Since $\alpha \notin \varphi[x_1, a[$ and $\beta \notin \varphi\,]b, a[$, the condition (1) holds.
   - The maximal directed path leading to a is unique, hence if a hyperedge E_i meets both A and C, it contains the interval $[b, a]$. Therefore, E_i has exactly $s-1$ colors from B, and has the color α on the vertex $a \in E_i \cap C$, complying with $2(c)$ and $2(a)$. Moreover, since the path $[x_1, a[$ is devoid of color α, the condition $2(b)$ automatically holds.

   Since all the conditions are satisfied, due to the Recoloring Lemma, the transposition of colors α and β on the set C yields a coloring φ' proper for the hypertree $\mathcal{T}$.

(4) If there is no vertex colored with α by the new coloring φ', then we have a proper $(k-1)$-coloring. In the other case let $\varphi := \varphi'$ and continue the procedure with step (1).

After a finite number of recolorings, the minimum distance between the root and the vertices with color α increases. Thus, eventually we obtain a proper $(k-1)$-coloring.

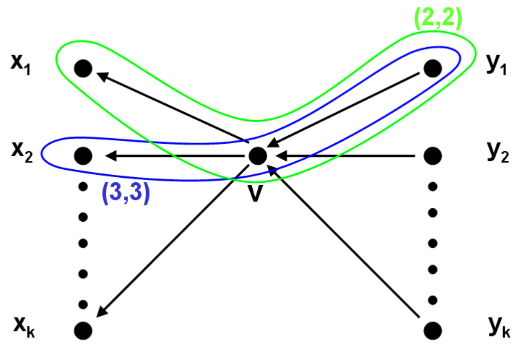

Figure 7: The structure of the DP-hypertree $\mathcal{T}_k^1$ in Example 2.

Therefore, if an RDP has a k-coloring for an integer $k > s$, then it also has a $(k-1)$-coloring. Taking into account further that $\chi(\mathcal{T}) \geq s$, these facts imply both the equation $\chi(\mathcal{T}) = s$ and the assertion that the chromatic spectrum has no gaps. □

Remark 13. *Due to the above proof, there exists an algorithm for color-bounded RDP hypertrees, which runs in $O(n^2)$ time (where n denotes the number of vertices) and generates a χ-coloring from any given k-coloring ($k > \chi = s$).*

In the remaining part of this section we show that there is a substantial difference between the behavior of RDP and DP hypertrees concerning lower chromatic number and feasible sets.

Proposition 23. *For every positive integer d there exists a 3-uniform color-bounded DP hypertree such that its lower chromatic number exceeds the value s by exactly d.*

Example 2. We construct the hypergraph $\mathcal{T}_k^1 = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ (where $k \geq 3$), satisfying the conditions of the proposition with $d = k - 2$, as follows. (The structure of this hypertree is illustrated on Figure 7.) The vertex set consists of $2k + 1$ vertices, $X = \{v, x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$, and there are two types of edges, $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$:

- $\mathcal{E}_1$: edges of the form $\{y_i, v, x_j\}$, with bounds $(3, 3)$ for every $1 \leq i \leq k$ and $1 \leq j \leq k$ with $i \neq j$.
- $\mathcal{E}_2$: edges of the form $\{y_i, v, x_i\}$, with bounds $(2, 2)$ for every $1 \leq i \leq k$.

We consider the directed host star with central vertex v, where each edge $\{y_i, v\}$ is oriented towards v and each edge $\{v, x_i\}$ is oriented away from v. Since every hyperedge is a directed path in this star, $\mathcal{T}_k^1$ is a DP hypertree.

Investigating the possible colorings of $\mathcal{T}_k^1$:

(1) For every x_i there exist $(3,3)$-edges in $\mathcal{E}_1$, implying that x_i has a different color from v; and, similarly, each y_i has a different color from v. Consequently, the $\{y_i, v, x_i\}$ edges from $\mathcal{E}_2$ can be colored with exactly two colors only if for each $1 \le i \le k$ the vertices x_i and y_i have the same color.

(2) Because of the $(3,3)$-edges, for all pairs of indices $i \neq j$, vertices y_i and x_j have different colors. Taking into consideration also (1), we get that the colors of vertices $v, x_1, x_2, \ldots, x_k$ are mutually different.

This means that in any coloring of $\mathcal{T}_k^1$, at least $k+1$ colors are used; that is, $\chi(\mathcal{T}_k^1) \ge (k+1)$. To show that $\mathcal{T}_k^1$ is indeed $(k+1)$-colorable, consider the color classes $\{v\}, \{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}$. As one can check, this determines a proper coloring for $\mathcal{T}_k^1$. (In fact this is the only one.)

For the 3-uniform DP hypertree $\mathcal{T}_k^1$, we have $s = 3$ and $\chi(\mathcal{T}_k^1) = k+1$, thus the considered difference $\chi - s$ is equal to $k - 2 = d$ for every $k \ge 3$. □

Note that, in order to increase the difference $\chi - s$ in DP hypertrees, we do not need an increasing value of s: it can be arbitrarily large under any fixed $s \ge 3$.

Concerning gaps in the chromatic spectrum, we prove:

Proposition 24. *For every positive integer g, there exists a color-bounded DP hypertree whose chromatic spectrum has a gap of size g.*

Example 3. We prove this proposition by an extended version of Example 2 as it can be seen on Figure 8. Let

$$\mathcal{T}_k^2 = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}), \qquad X = \{x_1, x_2, \ldots, x_k, v, y_1, y_2, \ldots, y_k, z_1, z_2, \ldots, z_k\}$$

and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$, where $\mathcal{E}_1$, $\mathcal{E}_2$ and their assigned values (s_i, t_i) are the same as in Example 2, whilst $\mathcal{E}_3$ contains the new edges $\{z_1, z_2, \ldots, z_k, v, x_i\}$ with bounds $(s_i, t_i) = (k+1, k+1)$ for every $1 \le i \le k$.

Obviously, this $\mathcal{T}_k^2$ is a DP hypertree and the effect of edges from $\mathcal{E}_1 \cup \mathcal{E}_2$ is the same as it was in the case of $\mathcal{T}_k^1$. Therefore, the properties (1) and (2) of $\mathcal{T}_k^1$ are valid for $\mathcal{T}_k^2$, too.

Considering a coloring φ of $\mathcal{T}_k^2$, there are two possibilities:

- If the vertices $z_1, z_2, \ldots, z_k, v$ have mutually different colors, then because of the $(k+1, k+1)$-edges from $\mathcal{E}_3$, each of $x_1, x_2, \ldots, x_k$ must have a common color with one of $z_1, z_2 \ldots, z_k, v$. According to (1), no new color can appear on the vertices $y_1, y_2, \ldots, y_k$. As a consequence, φ has to be a coloring with exactly $k+1$ colors. To prove that a $(k+1)$-coloring exists indeed, consider the color classes $\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}, \ldots, \{x_k, y_k, z_k\}, \{v\}$. This is clearly a proper coloring of $\mathcal{T}_k^2$.

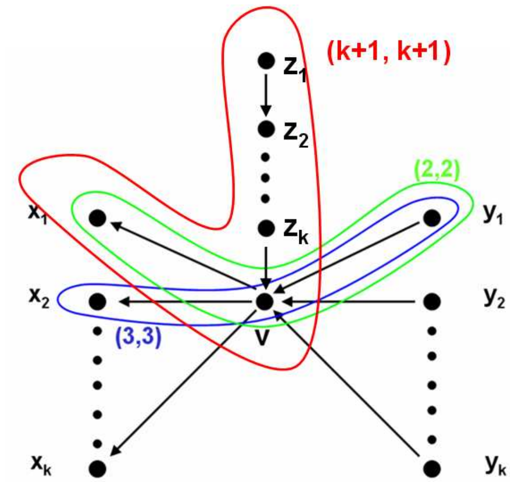

Figure 8: The structure of the DP-hypertree $\mathcal{T}_k^2$ in Example 3.

- If there are two vertices with a common color among $z_1, z_2, \ldots, z_k, v$, then according to the $(k+1, k+1)$-edges, each of the vertices $x_1, x_2, \ldots, x_k$ has a color different from the k colors of $z_1, z_2, \ldots, z_k, v$. Taking (2) into consideration, those are k different colors on the vertices $x_1, x_2, \ldots, x_k$. According to (1), no more new colors can appear on $y_1, y_2, \ldots, y_k$, hence precisely $2k$ colors occur on the whole $\mathcal{T}_k^2$. One of the possible $2k$-colorings has the following color classes: the singletons $\{z_1\}, \{z_2\}, \ldots, \{z_{k-1}\}$, the pair $\{z_k, v\}$, moreover $\{x_i, y_i\}$ for every $1 \leq i \leq k$.

Obviously, there is no more possibility for the coloring φ, thus the feasible set is $\Phi(\mathcal{T}_k^2) = \{k+1, 2k\}$ where the size of gap is $k - 2 = g$ for every $k \geq 3$. □

Note that in Examples 2 and 3 there are some redundant edges and a redundant vertex, too. For instance, the vertex y_k and the edges containing it may be canceled without changing the coloring properties. We have kept them, however, because in this way the description (and the argument, too) is simpler.

7.4 Hypertrees with unrestricted host trees

In this section we focus on color-bounded hypertrees in general. Despite still forming a quite restricted class of hypergraphs, it will turn out that they represent nearly all feasible sets belonging to color-bounded hypergraphs. Moreover, we prove that every feasible set of hypertrees occurs already in the class of 4-uniform hypertrees.

Theorem 16. *Let S be a finite set of positive integers. There exists a color-bounded hypertree $\mathcal{T}$ with feasible set $\Phi(\mathcal{T}) = S$ if and only if*

(*i*) $\min(S) = 1$ *or* $\min(S) = 2$*, and S contains all integers between* $\min(S)$ *and* $\max(S)$*, or*

(*ii*) $\min(S) \geq 3$.

Moreover, S is the feasible set of some r-uniform color-bounded hypertree, for an arbitrarily prescribed $r \geq 4$, if and only if it satisfies (*i*) *or* (*ii*).

This result will be proved at the end of this section. We first deal with the special case of *2-colorable* hypertrees. We are going to prove that their chromatic spectrum is necessarily gap-free, in sharp contrast to the entire class of color-bounded hypergraphs where every spectrum with $r_1 = 0$, $r_2 > 0$, and $r_i \geq 0$ ($i \geq 3$, with any finite number of nonzero values r_i) appears.

Theorem 17. *If a color-bounded hypertree is 2-colorable, then it has a gap-free chromatic spectrum.*

Proof If a hypertree $\mathcal{T}^*$ is 2-colorable, then $s \leq 2$. To prove the theorem, it is enough to show that from any k-coloring ($k \geq 4$) of the 2-colorable $\mathcal{T}^*$, a proper $(k-1)$-coloring can be created, too.

First, fix a host tree, and let two adjacent vertices of this host tree be contracted to one if they have the same color in the given k-coloring. After all such contractions we have a host tree T and a k-coloring φ of the contracted hypertree $\mathcal{T}$. Evidently, every proper coloring of $\mathcal{T}$ can be extended to a proper coloring of $\mathcal{T}^*$. Because of the contraction, φ colors any two neighboring vertices of T with different colors.

For a proper 2-coloring of a rooted tree (in the standard graph-theoretic sense) we shall use the term *alternate coloring*, or call the tree *alternately colored*. In this case the colors assigned to the root and to the neighbors of the root will be called the *first* and the *second* color, respectively. At this point we fix an arbitrary root vertex r in T and create an upper-root vertex r^* colored differently from r, but correspondingly to another vertex of T. We shall use also the terms *grandparent* and *grandchild* for 'parent of parent' and for the converse relation, respectively.

Now, we give a procedure that transforms the k-coloring φ to a $(k-1)$-coloring. Since there are at least 4 colors in the coloring φ, there exists a vertex colored differently from its grandparent. Choose a vertex with this property having the largest distance from the root of T, and let it be denoted by x_1, whilst the parent and grandparent of x_1 are denoted by y and z, respectively. Then determine all the children of y colored with $\varphi(x_1)$ and denote them by $x_1, x_2, \dots, x_j$. Due to the extremal choice of x_1, the subtrees $T(x_1), T(x_2), \dots, T(x_j)$ are alternately colored, and their second color is $\varphi(y)$.

To apply the Recoloring Lemma, we consider

$$C = \bigcup_{i=1}^{j} V(T(x_i)), \;\; B = \{y\}, \;\; A = X \backslash (B \cup C), \;\; \alpha = \varphi(x_1) = \ldots = \varphi(x_j), \;\; \beta = \varphi(z).$$

- Since $\alpha \neq \varphi(y) \neq \beta$, the condition (1) holds.
- If a hyperedge E_i meets both A and C, it surely contains at least one of the vertices $x_1, \dots, x_j$, hence $\alpha \in \varphi(E_i \cap C)$ complying with 2(a). In this case E_i also involves the vertex y, thus $|\,\varphi(E_i \cap B)| = 1 \geq s-1$; that is, 2($c$) holds.

  Moreover, every child of y has an alternately colored subtree with second color $\varphi(y) \neq \alpha$, hence α occurs in $T(y)$ only in the subtrees $T(x_1), T(x_2), \dots, T(x_j)$. Consequently, the color α cannot occur in $T(y) - C$. Hence, if $\alpha \in \varphi(E_i \cap A)$, then the hyperedge contains a vertex not belonging to $T(y)$, therefore $z \in E_i$ and $\beta \in \varphi(E_i)$ hold, ensuring 2(b).

Due to the Recoloring Lemma, a proper coloring is obtained by replacing the color α with β on the set C (since $\beta \notin \varphi(C)$), and the number of vertices having common color with their grandparent has increased. If it is a $(k-1)$-coloring (omitting the color α), then the procedure ends. Otherwise, the recoloring can be repeated, and the increasing number of vertices having common color with their grandparent assures that after a finite number of recolorings a $(k-1)$-coloring is obtained. This completes the proof. □

To investigate the feasible set and chromatic spectrum of hypertrees having no colorings with fewer than 3 colors, first we give a construction by which a connection is established between chromatic spectra of hypertrees and general hypergraphs.

Lemma 9. *For every color-bounded hypergraph $\mathcal{H}$ with chromatic spectrum* $(r_1, r_2, \dots, r_n)$*, there exists a color-bounded hypertree $\mathcal{T}$ whose chromatic spectrum is* $(p_1, p_2, \dots, p_{n+1})$*, where*

$$p_1 = 0 \qquad \text{and} \qquad p_{k+1} = r_k \quad \text{for all} \quad 1 \leq k \leq n.$$

Proof Consider a color-bounded hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ and transform it to a hypertree $\mathcal{T}$ involving a new central vertex v, in the following way:

$$\mathcal{T} = (X', \mathcal{E}', \boldsymbol{s}', \boldsymbol{t}'), \qquad X' = X \cup \{v\}, \qquad \mathcal{E}' = \mathcal{E}_1 \cup \mathcal{E}_2,$$

$\mathcal{E}_1 = \{\{x, v\} \mid x \in X\}$ and each of these edges has color-bounds (2,2),

$\mathcal{E}_2 = \{E_i \cup \{v\} \mid E_i \in \mathcal{E}\}$ where every edge $E_i \cup \{v\}$ has bounds $(s_i + 1, t_i + 1)$.

Forced by the edges from $\mathcal{E}_1$, the central vertex v determines a singleton color class in every proper coloring of $\mathcal{T}$. Removing this singleton from any color partition of $\mathcal{T}$, every hyperedge E_i has colors fewer by one as $E_i \cup \{v\}$ had, hence a proper color partition of $\mathcal{H}$ is obtained. Conversely, any proper color partition of $\mathcal{H}$ can be supplemented by the singleton $\{v\}$, yielding a proper partition for $\mathcal{T}$. Therefore, the proper k-partitions of $\mathcal{H}$ are in one-to-one correspondence with the proper $(k+1)$-partitions of $\mathcal{T}$; that is, $r_k = p_{k+1}$. Clearly, $p_1 = 0$ holds and there is a host star graph of $\mathcal{T}$ with central vertex v, consequently $\mathcal{T}$ satisfies the properties as required. □

The following theorem concerning the possible chromatic spectra of mixed hypergraphs has been proved in [32]:

(∗) Let $(r_1, r_2, \ldots, r_\ell)$ be any vector of non-negative integers such that $r_1 = 0$. Then there exists a mixed hypergraph whose chromatic spectrum is equal to $(r_1, r_2, \ldots, r_\ell)$.

Combining this result with Lemma 9 above, we obtain an immediate consequence for color-bounded hypertrees:

Corollary 14. *Every finite sequence $(r_1, r_2, \ldots, r_\ell)$ of nonnegative integers with $r_1 = r_2 = 0$ is the chromatic spectrum of some color-bounded hypertree.*

In contrast to this, the possible chromatic spectra of color-bounded hypertrees with $r_1 > 0$, or $r_1 = 0$ and $r_2 > 0$ have not yet been characterized.

Now we are in a position to complete the proof of the characterization theorem for feasible sets of color-bounded hypertrees.

Proof of Theorem 16 The necessity of conditions (i) and (ii) follows directly from Theorem 17 and from the fact that the chromatic spectrum of 1-colorable color-bounded hypergraphs is gap-free.

The sufficiency of (i) has been shown in Corollary 13, and one can find similar examples of 4-uniform interval hypergraphs, too. To verify the sufficiency of (ii), we take into consideration that every set S of positive integers omitting 1 is the feasible set of some (mixed) color-bounded hypergraph [29]. Then applying Lemma 9 it is proved that every S omitting 1 and 2 is a feasible set of some color-bounded hypertree.

In [32], for an arbitrarily given set S of integers at least 2, there was constructed a mixed hypergraph with feasible set S and with edges of sizes 2 and 3 only. To obtain a 4-uniform color-bounded hypertree from it, we first apply the construction from Lemma 9, and then supplement the constructed hypertree $\mathcal{T}$ with vertices v_1, v_2, v_3 and with a new edge $\{v, v_1, v_2, v_3\}$ having color-bounds $(1, 1)$. The edges containing only 2 or 3 vertices can be extended by some vertices of $\{v_1, v_2, v_3\}$, to contain exactly 4 vertices. Since this modification has no effect on the coloring properties of $\mathcal{T}$, every set S satisfying $\min(S) \geq 3$ can be obtained as a feasible set of some 4-uniform color-bounded hypertree.

A similar transformation — whose details are left to the reader — extends 4-uniform hypergraphs to r-uniform ones, for any $r \geq 5$ as well. This completes the proof of Theorem 16. □

7.5 Time complexity of the colorability of hypertrees

It was shown in Section 7.2 that 3-uniform color-bounded interval hypergraphs admit efficient coloring algorithm. On the other hand, since it is NP-complete to decide whether a 3-uniform mixed hypergraph is colorable [56], the constructive proof of Lemma 9 implies the same complexity for *4-uniform* color-bounded hypertrees (easily extendable to r-uniform ones with any $r \geq 4$).

In this section we prove that the hypertree colorability problem is hard already in the 3-uniform case.

Theorem 18. *The decision problem of colorability is* NP*-complete on 3-uniform color-bounded hypertrees.*

Proof We are going to apply a reduction from the classical problem of Hypergraph 2-Coloring restricted to 3-uniform hypergraphs, which is a well-known NP-complete problem [39].

First, we introduce the concepts of A-type and B-type subtrees for a given ordered 3-tuple (u, v, w) of vertices.

- The A-type subtree for (u, v, w) consists of the vertex set $\{y, e, u, v, w, f\}$ and of the following six edges:
  - $\{y, e, f\}$, with color-bounds $(3, 3)$,
  - $\{y, e, u\}, \{y, u, v\}, \{y, v, w\}, \{y, w, f\}$, each of them with bounds $(2, 2)$,
  - $\{y, u, w\}$, with bounds $(2, 3)$.

  Vertex y will be the fixed central vertex of the color-bounded hypertree to be constructed later; let us denote its color by 1.

  Consider a proper coloring φ of the above subtree. Due to the edge with bounds $(3, 3)$, the colors 1, $\varphi(e)$, and $\varphi(f)$ are mutually different. There must exist a vertex among u, v, w with color 1, otherwise the $(2, 2)$ edges would force that $\varphi(e) = \varphi(u) = \varphi(v) = \varphi(w) = \varphi(f)$, what contradicts the constraint $\varphi(e) \neq \varphi(f)$. On the other hand, two vertices from $\{u, v, w\}$ with color 1 would yield a monochromatic edge with y, but this is forbidden by the lower color-bound $s_i = 2$. Hence, there is exactly one vertex of u, v, w with color 1. Having chosen this vertex of color 1, the colors of the other two vertices are determined by the (2,2) edges. Namely, there are three possible proper color partitions for the A-type subtree of (u, v, w), determined by the position of $\varphi(y) = 1$ in $\{u, v, w\}$:
  - $\{y, u\}, \{e\}, \{v, w, f\}$,
  - $\{y, v\}, \{e, u\}, \{w, f\}$,
  - $\{y, w\}, \{e, u, v\}, \{f\}$.

The corresponding colorings will be termed A-colorings of (u, v, w).

- The B-type subtree for the 3-tuple (u, v, w) is determined by the three edges $\{y, u, v\}$, $\{y, v, w\}$, $\{y, w, u\}$, each of them assigned with the color-bounds $(2, 3)$. (In this case the ordering of (u, v, w) is irrelevant.)

  These triples specify that there exists at most one vertex among u, v, w having color $\varphi(y) = 1$. The corresponding colorings of the 3-tuple (u, v, w) will be called B-colorings.

Now, for every 3-uniform hypergraph $\mathcal{H} = (X, \mathcal{E})$, we construct a 3-uniform color-bounded hypertree $\mathcal{T}$ such that $\mathcal{T}$ is colorable if and only if $\mathcal{H}$ is 2-colorable.

Construction of $\mathcal{T}$ from $\mathcal{H}$:
Fix a new vertex y as the central vertex of $\mathcal{T}$, and then for every edge $E_i = \{a, b, c\}$ of $\mathcal{H}$ take 14 distinct new vertices $a_i, a'_i, b_i, b'_i, c_i, c'_i$, $e_i^1, e_i^2, e_i^3, e_i^4$, $f_i^1, f_i^2, f_i^3, f_i^4$. In order to force the A-coloring of the 3-tuples (a, a_i, a'_i), (b, b_i, b'_i), (c, c_i, c'_i) and (a_i, b_i, c_i), construct the A-type subtrees corresponding to them. (The last eight vertices e_i^j, f_i^j are used in these A-type subtrees, each vertex appearing in precisely one subtree.) Finally, let the hypertree be supplemented with edges corresponding to the B-type subtrees of the form (a'_i, b'_i, c'_i).

The 3-uniform hypergraph $\mathcal{T}$ obtained is a hypertree indeed, since the central vertex y is contained in each edge; i.e., the host tree can be chosen as a star. The vertices of $\mathcal{H}$ can belong to several $\mathcal{H}$-edges and to the same number of A-type subtrees of $\mathcal{T}$ accordingly, but the new vertices introduced at different $\mathcal{H}$-edges are all distinct.

If $\mathcal{H}$ is 2-colorable, then $\mathcal{T}$ is colorable:
For every fixed 2-coloring of an $\mathcal{H}$-edge $E_i = \{a, b, c\}$, we give an appropriate coloring of the respective A- and B-subtrees. We give the colors of vertices in tables according to the arrangement

$$\begin{pmatrix} a & b & c \\ a_i & b_i & c_i \\ a'_i & b'_i & c'_i \end{pmatrix} \qquad (\star)$$

Assume a coloring of $\mathcal{H}$ with colors 1 and 2. We assign the color 1 to y. If the colors of (a, b, c) are $(1, 1, 2)$, $(1, 2, 1)$, $(1, 2, 2)$, or $(2, 1, 2)$, then one possible combination of colors for the nine vertices is shown in the following tables:

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 3 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 2 \\ 3 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

The cases where the colors of (a, b, c) are $(2, 1, 1)$ or $(2, 2, 1)$, can be traced back by symmetry to the cases $(1, 1, 2)$ and $(1, 2, 2)$, respectively. (Because of the ordering, this reduction does not apply to $(1, 2, 1)$ and $(2, 1, 2)$.)

Having the coloring patterns on these triples, the colors for the vertices e_i^j and f_i^j can be chosen appropriately.

Consequently, any 2-coloring of $\mathcal{H}$ — assuming colors 1 and 2 — can be transformed to a 3-coloring of $\mathcal{T}$, where the color of y is 1, the vertices from $\mathcal{H}$ keep their original colors, and the new vertices are colored as shown in the tables above.

If $\mathcal{T}$ is colorable, then $\mathcal{H}$ is 2-colorable:
Assume a feasible coloring φ of $\mathcal{T}$, where the color of y is denoted by 1. For a subtree belonging to an edge E_i of $\mathcal{H}$, consider the table $(\star)$. Every column corresponds a 3-tuple A-colored by φ, hence there occurs exactly one vertex with color 1 in each column. Color 1 appears precisely once in the second row (A-coloring) and at most once in the third row (B-coloring). Thus, the first row (a, b, c) has one or two vertices with color 1. This holds for every E_i, therefore each edge of $\mathcal{H}$ has vertices from both types: colored with 1 and with another color by φ. Consequently, a feasible 2-coloring of $\mathcal{H}$ is obtained if we keep the color class 1 and replace all the other colors with color 2.

Since Hypergraph 2-Coloring is NP-complete, the two-way correspondence between the colorings of $\mathcal{H}$ and $\mathcal{T}$ described above implies that the colorability problem of 3-uniform color-bounded hypertrees is NP-complete, too. □

Remark 14.
By the construction above, we also obtain that the colorability problem of (3-uniform) color-bounded hypertrees remains NP*-complete even if the host tree is restricted to stars.*

7.6 Circular hypergraphs

In this section we investigate the coloring properties of circular color-bounded hypergraphs, a slight extension of interval hypergraphs. A hypergraph $\mathcal{H}$ is called *circular* if there exists a host cycle such that every hyperedge of $\mathcal{H}$ induces a connected subgraph (path or the entire cycle), so-called *arc*, on the host cycle. In the theory of *mixed* hypergraphs, this class has been studied e.g. in (Voloshin, Voss [61, 62]).

We assume a fixed positive direction around the cycle, and then the arc $[x, y]$ denotes the vertex set of the uniquely determined path leading from x to y in positive direction on the host cycle. We shall use the notation $]x, y]$, $[x, y[$, and $]x, y[$ for (half-) open paths analogously.

As it was proved in Section 7.2, for any colorable interval hypergraph $\mathcal{I}$ the lower chromatic number $\chi(\mathcal{I})$ is equal to $s = \max s_i$, and there is a polynomial-time algorithm to get a proper $\chi(\mathcal{I})$-coloring from an arbitrarily given proper coloring of $\mathcal{I}$.

We are going to show that the former property is not valid for circular hypergraphs; in fact, the difference $\chi - s$ can be arbitrarily large. But, on the other hand, assuming a fixed value of s, a sharp upper bound will be given for the lower chromatic number χ. We shall prove further that the feasible sets of circular hypergraphs are more restricted than it was in the case of hypertrees.

Proposition 25. *For every positive integer k, there exists a uniform circular color-bounded hypergraph $\mathcal{H}$ such that the difference $\chi(\mathcal{H}) - \max_{E_i \in \mathcal{E}} s_i$ is equal to k.*

Proof Given $k \in \mathbb{N}$, consider the circular hypergraph $\mathcal{H}$ on $n = 2k + 1$ vertices where the edge set contains all the arcs consisting of $k + 1$ consecutive vertices from the host cycle. We prescribe $s_i = t_i = k + 1$ colors on each hyperedge.

Since $n = 2k + 1$, any two vertices belong to a common $|E_i| = s_i$ edge and, consequently, they must get different colors. Thus, $\mathcal{H}$ is uniquely colorable with $2k + 1 = \chi(\mathcal{H})$ colors, and the difference is $\chi(\mathcal{H}) - \max s_i = k$, as claimed. □

Theorem 19. *If the circular color-bounded hypergraph $\mathcal{H}$ with $\max_{E_i \in \mathcal{E}} s_i = s$ is colorable, then the lower chromatic number $\chi(\mathcal{H})$ is at most $2s - 1$; and if the upper chromatic number is $\bar{\chi}(\mathcal{H}) \geq 2s - 1$, then there is no gap in the chromatic spectrum from $2s - 1$ to $\bar{\chi}(\mathcal{H})$.*

Proof The theorem is clearly valid for hypergraphs $\mathcal{H}$ such that $\bar{\chi}(\mathcal{H}) < 2s$ or $s = 1$. To prove it for $\bar{\chi}(\mathcal{H}) \geq 2s$ and $s \geq 2$, it will suffice to show that any k-coloring ($k \geq 2s$) of $\mathcal{H}$ can be transformed to a $(k-1)$-coloring. Then, starting from a $\bar{\chi}(\mathcal{H})$-coloring, one can apply this transformation $\bar{\chi}(\mathcal{H}) - 2s + 1$ times and reach a $(2s - 1)$-coloring via creating colorings one by one for each intermediate number of colors.

To reduce the number of colors from $k \geq 2s$ to $k - 1$, we present the following procedure.

Given a k-coloring φ of the hypergraph $\mathcal{H}$, let us choose two vertices a and b such that the arc $]a, b[$ is the longest one containing exactly $k - 1$ colors. Consequently, a and b have the same color α, which is the color omitted in $\varphi\,]a, b[$. If there exists only one vertex colored with α, we take $a = b$; in this case a recoloring without color α will be obtained.

Then we fix intervals $]a, a^*]$ and $[b^*, b[$, such that each of them contains precisely $s - 1$ colors. Since their union has at most $2s - 2 < k - 1$ colors, there exists a color β in $\varphi\,]a, b[$ that does not occur in $\varphi(]a, a^*] \cup [b^*, b[)$.

We are going to apply the Recoloring Lemma with

$$A =]a^*, b^*[, \quad B =]a, a^*] \cup [b^*, b[, \quad C = [b, a], \quad \alpha = \varphi(a) = \varphi(b) \quad \text{and} \quad \beta.$$

- Since $\varphi(B)$ is devoid of colors α and β, the condition (1) is satisfied.
- If a hyperedge E meets both A and C, it wholly involves at least one of the intervals $[a, a^*]$ and $[b^*, b]$, thus $\alpha \in \varphi(E \cap C)$ and $|\varphi(E \cap B)| \geq s - 1$ hold, complying with $2(a)$ and $2(c)$. Since $\alpha \notin \varphi(A)$, the condition $2(b)$ automatically holds.

By the Recoloring Lemma, the transposition of colors α and β on the arc C yields a proper coloring φ'. If φ' is a $(k-1)$-coloring, the algorithm stops, otherwise, the arc $]a, b[$ can be extended by at least two vertices such that it still has precisely $k-1$ different colors and the recoloring procedure can be repeated. In this way the longest arc of the host cycle with exactly $k-1$ colors gets extended in every recoloring. Hence, the algorithm yields a proper $(k-1)$-coloring of $\mathcal{H}$ after a finite number of steps. This completes the proof of the theorem. □

Note that the upper bound $2s-1$ is tight for the lower chromatic number of circular color-bounded hypergraphs, for any s, as shown by Proposition 25.

For $s \leq 2$, we get the following corollary:

Corollary 15. *If a circular color-bounded hypergraph is colorable and $s \leq 2$ holds, then the chromatic spectrum is gap-free and the lower chromatic number is at most 3.*

Proof We apply Theorem 19. If $s = 1$, then $2s - 1 = 1 = \chi(\mathcal{H})$, and obviously no gaps can occur. In the other case, $s = 2$ implies that $\chi(\mathcal{H}) = 2$ or $\chi(\mathcal{H}) = 3$, and that there cannot be a gap at any integer $k \geq 2s - 1 = 3$. Thus, the whole chromatic spectrum is gap-free. □

The condition $s \leq 2$ is valid for all circular *mixed* hypergraphs, therefore the previous corollary already implies

Corollary 16. *([35]) Every colorable circular mixed hypergraph $\mathcal{H}$ has a gap-free chromatic spectrum, and the lower chromatic number is at most 3.*

Remark 15. *¿From an algorithmic point of view, the above proof has the following consequence. For a generic input color-bounded circular hypergraph with a given k-coloring ($k > \chi$), a $(k-1)$-coloring can be determined in $O(n^2)$ time (where n denotes the number of vertices).*

8 Stably bounded hypergraphs: model comparison

The main goal of this chapter is to describe a unified framework for various concepts in the coloring theory of hypergraphs, and to study how some of its naturally arising subclasses are interrelated. The model presented here [6] includes, as particular cases, the proper (vertex) colorings in the classical sense, moreover the class of mixed hypergraphs, and also the color-bounded hypergraphs that have been introduced in Chapters 6 and 7.

While revising the manuscript of [6], we have learned that a subclass of stably bounded hypergraphs — prescribing only the upper monochromatic bound $\boldsymbol{b}$ — was studied previously in [50], [39] and [11], especially concerning approximation algorithms for the minimum number of colors in a proper coloring.

Although more general than all those, our present model with its four color-bound functions is still a subclass of the 'pattern hypergraphs' introduced by Dvořák et al. in [20], since in the latter the collection of feasible coloring patterns may be specified for each edge separately. Compared to that, however, our more restrictive conditions allow us to prove stronger results.

The essence of this model is that we can prescribe local constraints not only for the cardinality of the largest polychromatic subset, but also for that of the largest monochromatic one. That is, every hyperedge E_i is associated with four color-bounds s_i, t_i, a_i, and b_i. They prescribe that in a proper coloring the edge E_i has to get at least s_i and at most t_i different colors; and, on the other hand, there must exist a color occurring at least a_i times inside the edge, while there exists no color occurring more than b_i times.

This extension of the notion of color-bounded hypergraphs is reasonable from a theoretical point of view, since the new functions a_i and b_i are the monochromatic analogies to the earlier polychromatic bounds s_i and t_i. Furthermore, the extension has a strong practical motivation, too. Assigning types (colors) to the elements of a complex system (i.e., to the vertices), it is a quite frequent case that the constraints concern the number of occurrences of *fixed* types inside the hyperedges. For instance, we may have a condition that in a 12-element group E_i there should exist at least four elements labeled with type A, three or four elements should be labeled with type B, and the remaining vertices should be from types C and D. Although the basic definition of stably bounded hypergraphs contains no direct condition regarding the number of *fixed* types, we will see in Chapter 9 that the bounds a_i and b_i offer a possibility for a concise description. Consequently, stably bounded hypergraphs can be applied also for modeling these frequently appearing problems.

Conditions of the types $s_i = 1$, $t_i = |E_i|$, $a_i = 1$, and $b_i = |E_i|$ have no effect on the colorability properties of $\mathcal{H}$, because they are trivially satisfied in every coloring.

For this reason, we may restrict our attention to the subset of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$ that really means some conditions on at least one edge. We shall use Capital letters to indicate them. For instance, by an (S, T)-hypergraph we mean one where $a_i = 1$ and $b_i = |E_i|$ hold for all edges. In such hypergraphs it is usually the case — though not required by definition — that there is at least one edge $E_{i'}$ with $s_{i'} > 1$ and at least one edge $E_{i''}$ with $t_{i''} < |E_{i''}|$. Otherwise, e.g. if $s_i = 1$ also holds for all i, we may simply call it a T-hypergraph.

8.1 Small values and reductions

Here we point out some simple relations among the color-bound functions $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$. It will turn out that on 3-uniform hypergraphs without further restrictions, four different models are equivalent. On the other hand, for hypergraphs with arbitrary edge sizes, one of them is universal.

Proposition 26. *Let E_i be an edge in a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$. If $|E_i| \leq 3$, then for the largest cardinalities of polychromatic and monochromatic subsets of E_i, $\pi(E_i) + \mu(E_i) = |E_i| + 1$ holds.*

Proof It suffices to observe that any E_i has a unique partition into 1 or $|E_i|$ classes, verifying $\pi(E_i) + \mu(E_i) = |E_i| + 1$ for such trivial partitions; moreover, if $|E_i| = 3$, then the size distribution in precisely two nonempty partition classes is uniquely determined as $(2, 1)$, so that $\pi(E_i) = \mu(E_i) = 2$ in this case. □

Corollary 17. *Let $E_i \in \mathcal{E}$ be an edge with at most three vertices.*

1. *If $|E_i| = 1$, then $s_i = t_i = a_i = b_i = 1$ necessarily holds, and the edge may be deleted without changing the coloring properties of $\mathcal{H}$.*
2. *If $|E_i| = 2$, then between the local conditions the following equivalences are valid for $k = 1, 2$.*
   - *(i)* $s_i = k \Longleftrightarrow b_i = 3 - k$,
   - *(ii)* $a_i = k \Longleftrightarrow t_i = 3 - k$.
3. *If $|E_i| = 3$, then between the local conditions the following equivalences are valid for $k = 1, 2, 3$.*
   - *(i)* $s_i = k \Longleftrightarrow b_i = 4 - k$,
   - *(ii)* $a_i = k \Longleftrightarrow t_i = 4 - k$.

An important consequence is that, in the restricted class of 3-uniform hypergraphs, each pair in $(\boldsymbol{s}, \boldsymbol{b}) \times (\boldsymbol{t}, \boldsymbol{a})$ represents any nontrivial combination of $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$ in full generality:

Corollary 18. *If each edge of $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ has at most three vertices, then $\mathcal{H}$ has an equivalent description as an*

- *(S, T)-hypergraph,*
- *(S, A)-hypergraph,*
- *(T, B)-hypergraph,*
- *(A, B)-hypergraph.*

Proof Based on Corollary 17, every s-condition and a-condition can be transcribed to an equivalent b-condition and t-condition, respectively; and vice versa. □

The coincidences of conditions above do not carry over for edges with $|E_i| > 3$. Indeed, a 4-element set admits 2-partitions of both types $2 + 2$ and $3 + 1$ (and the situation is even worse for larger edges), hence there is no strict relation between $\pi(E_i)$ and $\mu(E_i)$ in either direction. Nevertheless, the following implications remain valid for edges of any size, by the pigeon-hole principle.

Proposition 27. *Let E_i be any edge in a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$. Then, between the conditions the following equivalences are valid.*

(i) $s_i = 2 \Longleftrightarrow b_i = |E_i| - 1,$

(ii) $a_i = 2 \Longleftrightarrow t_i = |E_i| - 1.$ □

In particular, for mixed hypergraphs we obtain

Corollary 19. *Every mixed hypergraph is a member of all the four classes of (S, T)-, (S, A)-, (T, B)-, and (A, B)-hypergraphs at the same time.*

Proof Every $\mathcal{C}$-edge E_i can be interpreted as $(s_i, a_i) = (1, 2)$, a $\mathcal{D}$-edge E_i corresponds to the bounds $(s_i, a_i) = (2, 1)$, whereas a bi-edge E_i is equivalent to the bounds $(s_i, a_i) = (2, 2)$. This yields membership in (S, A). Transcription to the other three models can be done via Proposition 27. □

Remark 16. *Contrary to mixed hypergraphs, in the general model the edges E_i of cardinality 2 with $t_i = 1$ or $a_i = 2$ usually cannot be contracted, despite their two vertices must get the same color in every proper coloring. The reason is that in $(\boldsymbol{a}, \boldsymbol{b})$ the multiplicities of colors are of essence. To keep track of them, one would need to introduce weighted vertices and interpret $(\boldsymbol{a}, \boldsymbol{b})$ as weighted conditions. We do not study weighted hypergraphs here.*

8.2 Class reductions and colorability

Some combinations between color-bound conditions can be done; moreover, some of their combinations always admit a proper coloring. We summarize these facts as follows.

Table 1

---

1. *Colorable pairs:*
   - (S, B)-hypergraphs allow every edge to be polychromatic, therefore the upper chromatic number equals the number of vertices.
   - (T, A)-hypergraphs allow every edge to be monochromatic, therefore the lower chromatic number equals 1.
2. *Combinations admitting uncolorability:*

   These are the sets of color-bound functions intersecting both (S, B) and (T, A); i.e., the minimal such sets are the pairs (S, T), (S, A), (A, B), and (T, B). The following operations show that all of them can be reduced to (S, A).
   - $b_i < |E_i|$: insert all $(b_i + 1)$-subsets of E_i with lower color-bound $s = 2$, and omit the condition b_i from E_i. This eliminates the function $\boldsymbol{b}$.
   - $t_i < |E_i|$: insert all $(t_i + 1)$-subsets of E_i with lower color-bound $a = 2$, and omit the condition t_i from E_i. This eliminates the function $\boldsymbol{t}$.
3. *Universal classes for colorability problems:*
   - S-hypergraphs [19] are universal models for n-colorable stably bounded structures where the question is to determine χ.
   - A-hypergraphs are universal models for 1-colorable stably bounded structures where the question is to determine $\overline{\chi}$.
   - (S, A)-hypergraphs are universal models for stably bounded structures where both χ and $\overline{\chi}$ are of interest.

Concerning feasible sets, the following assertions are valid.

Proposition 28. *If a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ has $\chi(\mathcal{H}) = 1$ or $\overline{\chi}(\mathcal{H}) = |X|$, then its chromatic spectrum is gap-free. Moreover, every interval of positive integers can be realized as the feasible set of hypergraphs with just one edge in each of the four types (S, T), (S, A), (T, B), and (A, B); and such a realization is possible even with an S-hypergraph and with a B-hypergraph.*

Proof If $\chi(\mathcal{H}) = 1$, then we have non-restrictive bounds $s_i = 1$ and $b_i = |E_i|$ for all edges E_i. Let φ be a k-coloring of $\mathcal{H}$ with $k > 1$. Taking the union of two arbitrarily chosen color classes of φ as just one new color class, no edge E_i will have smaller $\mu(E_i)$ or larger $\pi(E_i)$, hence a proper $(k-1)$-coloring is constructed. Starting from $k = \overline{\chi}(\mathcal{H})$, we obtain that all numbers of colors between 1 and $\overline{\chi}$ admit a proper coloring.

Similarly, for $\overline{\chi}(\mathcal{H}) = |X|$ we have $t_i = |E_i|$ and $a_i = 1$ for all E_i. If φ is a k-coloring of $\mathcal{H}$ with $k < |X|$, then some color class has more than one vertex, and splitting it into two nonempty classes in an arbitrary way we cannot violate the conditions s_i and b_i, so that a proper $(k+1)$-coloring is obtained. Starting from $k = \chi(\mathcal{H})$, all numbers of colors between χ and $|X|$ admit a proper coloring.

In order to construct a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with feasible set $\Phi(\mathcal{H}) = \{\ell \mid p \le \ell \le q\}$ for any given $q \ge p \ge 1$, we let $|X| = q$ that will ensure $\overline{\chi} = q$ for both S- and B-hypergraphs. To satisfy the equation $\chi = p$, we may simply assign $s = p$ to an edge whose cardinality is between p and q. In type B, this edge should have cardinality exactly p, assigned with the color-bound $b = 1$ that makes it polychromatic. □

Remark 17. *Conditions involving S are more flexible than those with B. Namely, for the types (S,T) and (S,A) we may take $\boldsymbol{s}(X) = p$ with any number n of vertices, because either of the conditions $\boldsymbol{t}(X) = q$ and $\boldsymbol{a}(X) = n - q + 1$ yields then $\overline{\chi} = q$, as the total number of colors cannot be larger than $n - \boldsymbol{a}(X) + 1$. On the other hand, concerning the hypergraph $\mathcal{H} = (X, \{X\})$ the only possible choice for $\boldsymbol{b}(X)$ is $\lceil n/p \rceil$, which guarantees $\chi = p$ if and only if $(p-1)\lceil n/p \rceil \le n - 1$. Thus, for (T,B) and (A,B) the set of feasible orders n is precisely $\bigcup_{b \ge 1} \{n \in \mathbb{N} \mid pb - b + 1 \le n \le pb\}$.*

Proposition 29. *For every finite sequence $(r_2, \ldots, r_k)$ of nonnegative integers with $r_k > 0$, there exist (S,T)-, (S,A)-, (A,B)-, and (T,B)-hypergraphs whose upper chromatic number is k and chromatic spectrum is $(r_1 = 0, r_2, \ldots, r_k)$.*

Proof As proved by Král' in [32], every spectrum $(r_2, \ldots, r_k)$ occurs in non-1-colorable (i.e., with $r_1 = 0$) mixed hypergraphs. Since every mixed hypergraph belongs to all of the four types by Corollary 19, the assertion follows. □

Hence, in hypergraphs $\mathcal{H}$ belonging to class types other than the trivially colorable ones which are subsets of $\{S, B\}$ and $\{T, A\}$, it remains a substantial question to determine the feasible set $\Phi(\mathcal{H})$. On the other hand, chromatic spectra and chromatic polynomials are of interest for trivially colorable classes, too.

8.3 Large gaps in the chromatic spectrum

Jiang et al. constructed in [29] a *mixed* hypergraph on $2k+4$ vertices and with a gap of size k in the chromatic spectrum, for all $k \ge 1$. This $2k+4$ is the smallest possible order, what follows from another result of the same paper (though this consequence is not formulated there explicitly).

In this section we extend this result by pointing out that the minimality of $2k+4$ for a gap of size k remains valid in the more general class of (T, A, B)-hypergraphs, too. In contrast to this, the exact minimum for (S, A)- and (S, T)-hypergraphs will be shown to be $k+5$. First, we prove an assertion that we shall use as a lemma but it can be of interest in itself, too. It contains, as subcases, all the types of (T, B)-, (A, B)-, and mixed hypergraphs.

Proposition 30. *If a (T, A, B)-hypergraph on n vertices has a gap at g, then its lower chromatic number χ is at least $2g - n + 2$.*

Proof By definition, there exists an integer $j \geq g + 1$ such that the hypergraph has a coloring φ with exactly j colors but there is no proper $(j-1)$-coloring.

Suppose first that $2j - 2 \geq n$. Then there occur at least $2j - n \geq 2$ singleton color classes in φ. Considering two of them, say $\{x\}$ and $\{y\}$, their union yields a non-feasible $(j-1)$-coloring. After the identification of $\varphi(x)$ and $\varphi(y)$, however, all the bounds t_i and a_i remain fulfilled. Consequently, the obtained $(j-1)$-coloring can be non-feasible only because of an edge E_i containing both vertices x and y and having bound $b_i = 1$. Thus, x and y must have different colors in every feasible coloring. For the same reason, any two vertices from the at least $2j - n$ singletons are differently colored in any χ-coloring, too. This implies $\chi \geq 2j - n$. Due to the condition $j \geq g + 1$, the inequality $\chi \geq 2g - n + 2$ follows.

On the other hand, if $2j - 2 < n$, considering the gap at $g \leq j - 1$ we obtain the upper bound $2g - n + 2 \leq 2j - n \leq 1$, thus the inequality $\chi \geq 2g - n + 2$ automatically holds. □

We mention the following consequence that was proved for *mixed* hypergraphs in [29]. Tightness follows from a construction of the same paper.

Corollary 20. *If a (T, A, B)-hypergraph is ℓ-colorable and has a gap at g, then it has at least $2g + 2 - \ell$ vertices.*

Theorem 20. *If a (T, A, B)-hypergraph has a gap of size $k \geq 1$ in its chromatic spectrum, then it has at least $2k + 4$ vertices. Moreover, this bound is sharp; that is, for every positive integer k there exist mixed, (T, B)- and (A, B)-hypergraphs on $|X| = 2k + 4$ vertices, whose chromatic spectrum has a gap of size k.*

Proof Suppose that a (T, A, B)-hypergraph has an ℓ-coloring and an $(\ell + k + 1)$-coloring, but all integers in between are gaps. Then we can apply Proposition 30 with $g = \ell + k$, so that $2\ell + 2k - n + 2 \leq \chi \leq \ell$ is obtained. Moreover, every 1-colorable hypergraph has continuous chromatic spectrum, hence $\ell \geq 2$ holds and the above facts imply that the lower bound $n \geq \ell + 2k + 2 \geq 2k + 4$ is valid.

To show that the bound is sharp, we consider the construction from [29]. The hypergraph $\mathcal{H}_{2,k+3}$ is defined on the $(2k+4)$-element vertex set $\{x_1, x_2, a_1, a_2, \ldots, a_{k+1}, b_1, b_2, \ldots, b_{k+1}\}$, with the following edges:

- Triples of the form $\{x_i, a_j, b_j\}$ for $i = 1, 2$ and for all $1 \le j \le k+1$. They are bi-edges in the mixed hypergraph and have bounds $(t, b) = (2, 2)$ or $(a, b) = (2, 2)$ in the other models.

- Quadruples of the form $\{a_i, a_j, b_i, b_j\}$ for all $1 \le i < j \le k+1$, as $\mathcal{D}$-edges, with bounds $(t, b) = (4, 3)$ or $(a, b) = (1, 3)$.

- Triples of the form $\{a_i, a_j, b_i\}$ and $\{a_i, b_i, b_j\}$ for any two distinct indices $i, j \in \{1, 2, \ldots, k+1\}$. They are $\mathcal{C}$-edges or equivalently have bounds $(t, b) = (2, 3)$ and $(a, b) = (2, 3)$, respectively.

- The pair $\{x_1, x_2\}$ as a $\mathcal{D}$-edge, with bounds $(t, b) = (2, 1)$ or $(a, b) = (1, 1)$.

The feasible set of this hypergraph is $\{2, k+3\}$, as it was proved in [29]. This fact remains valid in all of the three models considered, thus the assertion follows. □

In [4] we proved that (S, T)-hypergraphs can have a gap of size k only if the number of vertices is at least $k+5$, and this bound is tight. Now, we extend the lower bound of this result to all stably bounded hypergraphs, and show that it is tight already for (S, A)-hypergraphs.

Theorem 21. *If a stably bounded hypergraph has a gap of size $k \ge 1$ in its chromatic spectrum, then it has at least $k+5$ vertices. Moreover, this estimate is sharp, already for the types (S, A) and (S, T); that is, for every positive integer k there exist (S, A)- and (S, T)-hypergraphs on $|X| = k+5$ vertices, whose chromatic spectrum has a gap of size k.*

Proof We have proved in Proposition 28 that if a stably bounded hypergraph has a 1-coloring or a totally polychromatic n-coloring, then its chromatic spectrum is gap-free. Hence, the only possibility for having a gap of size $k \ge 1$ on fewer than $k+5$ vertices would be with $n = k+4$ and with the feasible set $\{2, k+3\}$.

Assume for a contradiction that this is the case. Because of 2-colorability, the inequality $s_i \le 2$ is valid for every edge E_i. Let now φ be a coloring with precisely $k+3 = n-1$ colors. Then $k+2$ of the color classes (i.e., all but one) in φ are singletons. Taking the union of two arbitrarily chosen 1-element classes, say $\{x\}$ and $\{y\}$, we get a non-feasible color partition. This change can never decrease the size of monochromatic subsets or increase the number of distinct colors occurring inside any edge, therefore all of the bounds a_i and t_i are kept satisfied.

Hence, there exists an edge E_i for which either the bound s_i (what is at most two) gets violated, or its monochromatic subset becomes larger than b_i. The former means, however, that E_i becomes monochromatic. That is, we have $s_i = 2$ and $E_i = \{x, y\}$, hence x and y are colored differently in every feasible coloring. On the other hand, since $\varphi(x) \ne \varphi(y)$, in the modified coloring the color of $\{x, y\}$ does not occur on any other vertex. Hence, if b_i gets violated, then $b_i = 1$ must hold, and again we can conclude that x and y are colored differently in every feasible coloring.

This property is valid for any two of the $k+2 \geq 3$ singletons, what contradicts the assumption that the hypergraph is 2-colorable. Hence, there cannot exist any stably bounded hypergraphs with a gap of size k on fewer than $k+5$ vertices.

To show that a gap of size k is realizable on $k+5$ vertices, we present a hypergraph with feasible set $\{3, k+4\}$ on $k+5$ vertices, for every positive integer k. This hypergraph can be interpreted as (S,T)- and (S,A)-hypergraph as well.

Example 4. Consider the hypergraph $\mathcal{H}_k$ with vertex set $X = \{x_1, x_2, y_1, y_2, \ldots, y_{k+3}\}$ and with edge set $\mathcal{E} = \{\{x_1, x_2, y_i, y_j\} \mid 1 \leq i < j \leq k+3\}$, where each of the edges has bounds $(s,t) = (3,3)$ and $(s,a) = (3,2)$ in models (S,T) and (S,A), respectively. Since each hyperedge contains four vertices, the above bounds force that every hyperedge has to have exactly three different colors.

If a proper coloring φ assigns different colors to x_1 and x_2, there appears exactly one more color on the vertices $y_1, y_2, \ldots, y_{k+3}$. It is clearly realizable with color classes $\{x_1\}$, $\{x_2\}$, and $\{y_1, y_2, \ldots, y_{k+3}\}$, hence $\mathcal{H}$ can be colored only with precisely three colors in this case.

On the other hand, if φ assigns the same color to x_1 and x_2, the $(3,3)$-edges are properly colored if and only if any two distinct vertices y_i and y_j have colors different from each other and from $\varphi(x_1)$, too. Thus, in this case we obtain a proper $(k+4)$-coloring.

Since there are no other cases, the feasible set is $\{3, k+4\}$; that is, for every $k \geq 1$ the hypergraph $\mathcal{H}_k$ has a gap of size k on $k+5$ vertices. □

Theorems 20 and 21 together characterize the minimum order of a hypergraph of any nontrivial type for a gap of size k: the minimum is $k+5$ if and only if the type contains (S,T) or (S,A), and it is $2k+4$ if and only if it does not contain S but contains B and at least one of T and A. In any other case, the spectrum is gap-free.

8.4 Comparison of the sets of chromatic polynomials

We have already seen that any type of nontrivial combinations of $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$ can be expressed with $(\boldsymbol{s}, \boldsymbol{a})$ on applying part 2 of Table 1, if no structural conditions are imposed; and, furthermore, for 3-uniform hypergraphs each pair in $(\boldsymbol{s}, \boldsymbol{b}) \times (\boldsymbol{t}, \boldsymbol{a})$ would work equally nicely. Here we prove that this latter equivalence is not valid in general.

To formulate observations providing a more detailed information, let us denote by $\mathcal{P}_{X,Y}$ and $\mathcal{P}_X$ the sets of chromatic polynomials belonging to the classes of hypergraphs of type (X,Y) and of type X, respectively, for any $X, Y \in \{S,T,A,B\}$. Similarly, the set of chromatic polynomials appearing in the case of mixed hypergraphs will be denoted by $\mathcal{P}_m$.

Theorem 22. *For the sets of chromatic polynomials belonging to* (S, A)*-*, (A, B)*-*, (S, T)*-*, (T, B)*-, and mixed hypergraphs, the relations* $\mathcal{P}_{S,A} = \mathcal{P}_{A,B} \supsetneq \mathcal{P}_{S,T} = \mathcal{P}_{T,B} = \mathcal{P}_m$ *hold.*

Proof

1. According to Corollary 19, each mixed hypergraph has a chromatic equivalent in each of those four stably bounded subclasses. Thus, the set $\mathcal{P}_m$ is contained in each of $\mathcal{P}_{S,A}$, $\mathcal{P}_{A,B}$, $\mathcal{P}_{S,T}$, and $\mathcal{P}_{T,B}$.

2. On the other hand, as it has been shown, the bound $b_i < |E_i|$ can be replaced by some (b_i+1)-element $\mathcal{D}$-edges, whilst the elimination of the bound $t_i < |E_i|$ can be done by inserting some $(t_i + 1)$-element $\mathcal{C}$-edges. Therefore, every (T, B)-hypergraph has a chromatically equivalent mixed hypergraph (on the same vertex set). Taking into consideration the observation 1, the equality of $\mathcal{P}_{T,B}$ and $\mathcal{P}_m$ is obtained.

3. By Corollary 12 the sets $\mathcal{P}_{S,T}$ and $\mathcal{P}_m$ are equal. It worth noting that there exists an (S, T)-hypergraph with no chromatic equivalent mixed hypergraph on the same number of vertices. For instance, due to Theorem 21, there exists an (S, T)-hypergraph with a gap of size 2 on seven vertices, whilst in the case of mixed hypergraphs it needs at least eight vertices, by a result of [29].

4. By the elimination of $\boldsymbol{t}$, the (S, T)-hypergraphs can be modeled in (S, A), hence $\mathcal{P}_{S,A} \supseteq \mathcal{P}_{S,T}$. We are going to show that the sets of chromatic spectra, and consequently also the chromatic polynomials, of (S, A)- and (S, T)-hypergraphs are not equal.

   Let $\mathcal{H}_{s,a}$ have four vertices and just one 4-element edge with bounds $a = 3$ and $s = 1$. Obviously, $r_1 = 1$ and $r_2 = 4$. On the other hand, it was proved in Section 6.3 that in 1-colorable (S, T)-hypergraphs the value of r_2 always is of the form $2^{n-1} - 1$. Since this property is not valid for $\mathcal{H}_{s,a}$, it cannot have a chromatically equivalent (S, T)-hypergraph.

5. Since any chromatic spectrum with $r_1 = 0$ belongs to some mixed hypergraphs, the same holds for (S, A)- and (A, B)-hypergraphs, too. Thus, a difference between $\mathcal{P}_{S,A}$ and $\mathcal{P}_{A,B}$ might occur only on hypergraphs with $r_1 = 1$. The assumption of 1-colorability in an (S, A)-hypergraph implies that every edge E_i has bounds $(s_i, a_i) = (1, a_i)$, whereas in an (A, B)-hypergraph it implies $(a_i, b_i) = (a_i, |E_i|)$ for every edge. These two color-bound conditions clearly are equivalent on each edge. Hence, the possible chromatic spectra and consequently the chromatic polynomials are the same: $\mathcal{P}_{S,A} = \mathcal{P}_{A,B}$.

   Nevertheless, there exist some (S, A)-hypergraphs not having chromatic equivalent (A, B)-hypergraphs on the same number of vertices. Similarly to the example in step 4 of the proof, one can see that there exists an (S, A)-hypergraph

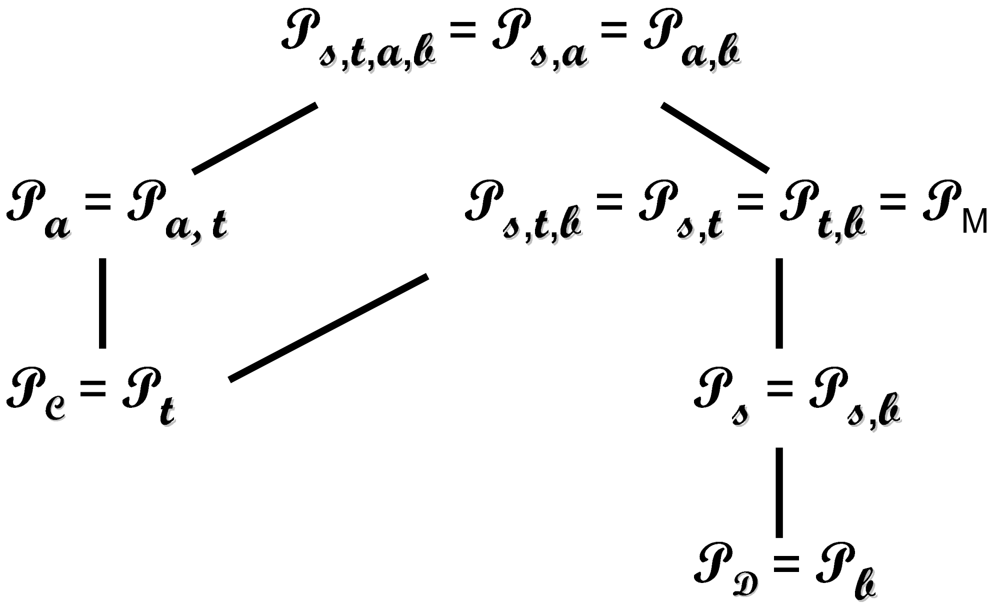

Figure 9: Hasse-diagram of possible chromatic polynomials belonging to different structure classes

on seven vertices with feasible set $\{3,6\}$, but to generate this feasible set in (A,B)-hypergraphs needs at least eight (in fact, at least nine) vertices. □

As regards modeling with the same number of vertices, the previous proof yields the following observation.

Remark 18. *Every mixed hypergraph has a chromatically equivalent (T,B)-hypergraph such that their vertex sets are of the same cardinality, and vice versa. This stronger condition does not hold for any other pairs of the models listed above.*

We close this subsection with supplements of Theorem 22 regarding other types of stably bounded hypergraphs.

Proposition 31. *Concerning the possible chromatic polynomials of S-, T-, A-, B-, $\mathcal{C}$- ('mixed', without $\mathcal{D}$-edges) and $\mathcal{D}$- (classical) hypergraphs the following relations hold:*

1. $\mathcal{P}_m \supsetneqq \mathcal{P}_S = \mathcal{P}_{S,B} \supsetneqq \mathcal{P}_{\mathcal{D}} = \mathcal{P}_B$,
2. $\mathcal{P}_{S,A} \supsetneqq \mathcal{P}_A = \mathcal{P}_{A,T} \supsetneqq \mathcal{P}_{\mathcal{C}} = \mathcal{P}_T$,
3. $\mathcal{P}_m$ *and* $\mathcal{P}_A$ *are incomparable.*

Proof

1. Every $\mathcal{D}$-edge E_i can be interpreted equivalently with bound $b_i = |E_i| - 1$, whilst a bound $b_i < |E_i|$ can be replaced by some $(b_i + 1)$-element $\mathcal{D}$-edges, therefore $\mathcal{P}_\mathcal{D} = \mathcal{P}_B$ holds.

   Every $\mathcal{D}$-edge evidently means an edge with bound $s = 2$, hence $\mathcal{P}_\mathcal{D} \subseteq \mathcal{P}_S$ is clear. On the other hand, let us consider the S-hypergraph $\mathcal{H} = (X, \{X\}, \boldsymbol{s})$ with $|X| = 5$ vertices and with color-bound $\boldsymbol{s}(X) = 3$. Its chromatic spectrum is $(0, 0, 25, 10, 1)$. Assuming a $\mathcal{D}$-hypergraph with this spectrum, it should have five vertices and each of its 3-partitions should yield a proper coloring. In particular, for any three vertices there should exist a coloring where they get the same color, implying that there can occur $\mathcal{D}$-edges only of sizes 4 and 5. Consequently, the 2-partitions with color classes of size 2 and 3 are not forbidden, what contradicts $r_2 = 0$. Therefore, this S-hypergraph has no equivalent $\mathcal{D}$-hypergraph, implying $\mathcal{P}_S \supsetneqq \mathcal{P}_\mathcal{D}$.

   By the elimination of $\boldsymbol{b}$, we can transform the structures of type (S, B) to type S, hence $\mathcal{P}_S = \mathcal{P}_{S,B}$. It is also clear that $\mathcal{P}_S \subseteq \mathcal{P}_{S,T} = \mathcal{P}_m$, and that mixed hypergraphs having gaps in their chromatic spectra cannot be modeled in S-hypergraphs. That is, $\mathcal{P}_S \subsetneqq \mathcal{P}_m$ is obtained.

2. Any $\mathcal{C}$-edge E_i can be considered as an edge with bound $t_i = |E_i| - 1$, whilst any bound $t_i < |E_i|$ can be expressed by $\mathcal{C}$-edges, hence $\mathcal{P}_\mathcal{C} = \mathcal{P}_T$.

   By eliminating $\boldsymbol{t}$, every (A, T)-hypergraph can be rewritten only with the bound $\boldsymbol{a}$, thus $\mathcal{P}_A = \mathcal{P}_{A,T}$. Moreover $\mathcal{P}_{S,A} \supseteq \mathcal{P}_A$ trivially holds.

   To show that there exist A-hypergraphs having no chromatically equivalent $\mathcal{C}$-hypergraphs, we recall the example from step 4 in the proof of Theorem 22. This (S, A)-hypergraph can be considered as just an A-hypergraph, and since it has no equivalent of type (S, T), the same is true for mixed- and $\mathcal{C}$-hypergraphs, too. Consequently, $\mathcal{P}_A \supsetneqq \mathcal{P}_\mathcal{C}$, and because of the 1-colorability of every A-hypergraph, $\mathcal{P}_{S,A} \neq \mathcal{P}_A$ is valid as well.

3. By the previous example there exist A-hypergraphs that have no equivalent mixed hypergraphs whereas mixed hypergraphs admitting no 1-coloring cannot be equivalent to any A-hypergraphs. □

Proposition 32. *Concerning the possible chromatic polynomials of stably bounded hypergraphs involving at least three types of conditions, the following equations hold:*

1. $\mathcal{P}_{S,T,A,B} = \mathcal{P}_{S,A,B} = \mathcal{P}_{S,A,T} = \mathcal{P}_{S,A}$,

2. $\mathcal{P}_{A,B,T} = \mathcal{P}_{A,B} = \mathcal{P}_{S,A}$,

3. $\mathcal{P}_{S,T,B} = \mathcal{P}_{S,T}$.

Proof The reductions described in part 2 of Table 1 yield:

- The color-bound function $\boldsymbol{t}$ can be expressed by the function $\boldsymbol{a}$. Consequently, if a type contains T and A together, then omitting T the set of possible chromatic polynomials does not change.
- Similarly, the function $\boldsymbol{b}$ can be reduced to $\boldsymbol{s}$, therefore in the presence of S the cancelation of B cannot make a change in the set of possible chromatic polynomials.

These observations immediately imply the statements listed above, except for the last equation in part 2, what has been proved in Theorem 22. □

8.5 Complexity of testing colorability

In this section we investigate the time complexity of the following two algorithmic problems.

Colorability

Instance: A hypergraph $\mathcal{H}$ of a given type.
Question: Is $\mathcal{H}$ colorable?

Unique k-Colorability

Instance: A hypergraph $\mathcal{H}$ of a given type, together with a proper k-coloring φ.
Question: Does $\mathcal{H}$ admit any proper coloring other than φ?

For the former, we simply extend the NP-hardness result of Chapter **??** from (S,T)-hypergraphs to all nontrivial combinations of the color-bound functions. On the other hand, the situation with the latter problem is more interesting. We choose the value $k = n - 1$ and prove that two of the non-trivial pairs, namely those containing S, lead to intractability; but the other two, containing B, admit a good characterization and polynomial-time algorithms.

In general, it should be noted that Colorability clearly belongs to NP, whereas Unique k-Colorability is in co-NP. Moreover, since a hypergraph on n vertices cannot have more than $\binom{n}{2}$ proper $(n-1)$-colorings, we can see that for $k = n - 1$ (and also if k is as large as n minus a constant) it does not change the complexity status of the problem if a k-coloring is not given in the input.

8.5.1 Colorability of 3-uniform hypergraphs

It was first observed in [56] that the recognition problem of colorable mixed hypergraphs is NP-complete in general, and also when restricted to 3-uniform mixed hypergraphs. There are some important classes with a nice structure, however, that admit efficient algorithms. In [54] a simple necessary and sufficient condition was given for the colorability of *mixed hypertrees*, from which an efficient algorithm is obtained, too.

On the other hand, we have shown in Section 7.5 that the colorability of 3-uniform (S,T)-hypertrees is NP-complete. We have also seen in Corollary 18 that every 3-uniform stably bounded hypergraph has equivalent representations with all the types of (S,T)-, (S,A)-, (T,B)-, and (A,B)-hypergraphs, and those can be constructed in linear time. In this way, an input of any of these types can efficiently be transformed to an (S,T)-hypergraph. Consequently, the result of Theorem 18 can be extended as follows.

Theorem 23. *The* Colorability *problem is* NP*-complete on each of the following classes of hypergraphs:*

- *3-uniform* (S,T)*-hypertrees,*
- *3-uniform* (S,A)*-hypertrees,*
- *3-uniform* (T,B)*-hypertrees,*
- *3-uniform* (A,B)*-hypertrees.*

It is worth comparing Theorem 23 with the following results: there are linear-time algorithms for deciding whether a *mixed* hypertree is colorable, and also for finding a proper coloring if there exists one [54], whereas determining the upper chromatic number of a mixed hypertree without edges larger than three is NP-complete [34].

Let us note further that NP-completeness remains valid if we assume that the host tree is a star. On the other hand, it can be proved that 3-uniform stably bounded *interval* hypergraphs admit a linear-time colorability test and a linear-time coloring algorithm, too.

8.5.2 Uniquely $(n-1)$-colorable (S,T)- and (S,A)-hypergraphs

Although it is hard to test whether an unrestricted mixed hypergraph is uniquely colorable [56], this is not the case if $\overline{\chi}$ is very large. For the latter case, Niculitsa and Voss [45] described a characterization of uniquely $(n-1)$-colorable, and also of uniquely $(n-2)$-colorable mixed hypergraphs.

In sharp contrast to this, our Theorem 12 states that the recognition of uniquely $(n-1)$-colorable (S,T)-hypergraphs is hard. Here we show how the construction can be extended to (S,A)-hypergraphs.

Theorem 24. *The* UNIQUE $(n-1)$-COLORABILITY *problem is co-*NP*-complete on* (S,A)*-hypergraphs.*

Proof As we have already mentioned, membership in co-NP is clear. To prove hardness, let us recall from the proof of Theorem 12 the reduction for (S,T)-hypergraphs, from the problem of determining the *chromatic number of Steiner triple systems*[10] Phelps and Rödl proved in [47] that it is NP-complete to decide whether a Steiner triple system — viewed as a 3-uniform $\mathcal{D}$- (classical) hypergraph — is colorable with 14 colors. Given an input Steiner triple system $\mathcal{S} = STS(n-2) = (X,\mathcal{B})$ of order $n-2$ with vertex set $X = \{x_1,\ldots,x_{n-2}\}$ and edge set $\mathcal{B}$, an (S,T)-hypergraph $\mathcal{H} = (X', \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ is constructed as follows. We set $X' = X \cup \{z_1, z_2\}$, where z_1, z_2 are two new vertices, and consider the following edges with respective color-bounds:

- $B' = B \cup \{z_1, z_2\}$ with $\boldsymbol{s}(B') = 4$ and $\boldsymbol{t}(B') = 5$, for all blocks $B \in \mathcal{B}$;
- $W' = W \cup \{z_1, z_2\}$ with $\boldsymbol{s}(W') = 1$ and $\boldsymbol{t}(W') = 16$, for all 15-element subsets of X;
- $E_{i,j} = \{x_i, z_j\}$ with $\boldsymbol{s}(E_{i,j}) = \boldsymbol{t}(E_{i,j}) = 2$, for all $1 \le i \le n-2$ and $j = 1, 2$.

In this (S,T)-hypergraph, every t_i is either $|E_i|$ or $|E_i|-1$. Hence, it is easy to eliminate $\boldsymbol{t}$ along the lines of Proposition 27 and obtain an equivalent (S,A)-hypergraph: we simply define

$$\boldsymbol{a}(B') = 1, \qquad \boldsymbol{a}(W') = 2, \qquad \boldsymbol{a}(E_{i,j}) = 1$$

for all edges $B', W', E_{i,j} \in \mathcal{E}$. From the argument in Section 6.4 it follows that $\mathcal{H}$ is *not* uniquely $(n-1)$-colorable if and only if $\mathcal{S}$ has a proper coloring with at most 14 colors; and certainly the same holds for the derived (S,A)-hypergraph, too. Thus, co-NP-hardness follows. □

8.5.3 Uniquely $(n-1)$-colorable (T,B)- and (A,B)-hypergraphs

In this subsection we characterize the uniquely $(n-1)$-colorable (T,B)- and (A,B)-hypergraphs. In the models (S,A) and (S,T) studied in the previous subsection, the decision problem of unique $(n-1)$-colorability was proved to be co-NP-complete. In contrast to this, the characterization presented below yields polynomial-time algorithms for (T,B)- and (A,B)-hypergraphs.

[10] Let us recall the definition: a Steiner triple system (STS) of order n is an n-element set X together with a set $\mathcal{B}$ of 3-element subsets of X (called blocks) with the property that each 2-element subset of X is contained in exactly one block.

Before the characterization, let some terminology be introduced:

- $\{x, y\}$ is called $\mathcal{B}_1$-edge if x and y are contained in a common hyperedge E_i having bound $b_i = 1$. (This corresponds to a graph-edge in the usual sense.)
- E_i is called $\mathcal{B}_2$-edge if $b_i = 2$.
- E_i is called $\mathcal{C}$-edge if $t_i = |E_i| - 1$.

Concerning a given (T, B)-hypergraph, the set of $\mathcal{C}$-, $\mathcal{B}_1$- and $\mathcal{B}_2$-edges will be denoted by $\mathcal{C}$, $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. As a side-product of the characterization theorem, it will turn out that if an edge has bound $b_i \geq 3$, then the exact value of b_i has no influence on unique $(n-1)$-colorability.

Theorem 25. *A (T, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{t}, \boldsymbol{b})$ on $|X| = n$ vertices is uniquely $(n-1)$-colorable if and only if the following conditions hold:*

(α) $\max_{E_i \in \mathcal{E}}(|E_i| - t_i) = 1$.

(β) *The set $C^* := \bigcap_{C \in \mathcal{C}} C$ contains at least two vertices and induces a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge.*

Moreover, denoting by y_1 and y_2 the vertices of the missing $\mathcal{B}_1$-edge,

(γ) *$X \setminus \{y_1, y_2\}$ is a complete $\mathcal{B}_1$-graph.*

(δ) *For each vertex $x \in X \setminus C^*$, at least one of the relations $\{x, y_1\} \in \mathcal{B}_1$, $\{x, y_2\} \in \mathcal{B}_1$ and $\{x, y_1, y_2\} \subseteq E_i \in \mathcal{B}_2$ holds.*

(ϵ) *For each pair of vertices $x_j, x_k \in X \setminus C^*$, if $\{x_j, x_k\}$ intersects every $\mathcal{C}$-edge, then either there exist $\mathcal{B}_1$-edges $\{z, x_j\}$ and $\{z, x_k\}$ for a $z \in \{y_1, y_2\}$, or there exist $\mathcal{B}_1$-edges $\{z, y_1\}$ and $\{z, y_2\}$ for a $z \in \{x_j, x_k\}$.*

Proof Consider a uniquely $(n-1)$-colorable (T, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{t}, \boldsymbol{b})$. Since it admits an $(n-1)$-coloring, where each edge has at least $|E_i| - 1$ colors, $t_i \geq |E_i| - 1$ holds. On the other hand, since the n-coloring is not feasible, there is some hyperedge with bound $t_i = |E_i| - 1$. Consequently, we have $\max_{E_i \in \mathcal{E}}(|E_i| - t_i) = 1$, according to ($\alpha$).

Let φ be a proper $(n-1)$-coloring, and assume without loss of generality that its color classes are $\{x_1\}, \{x_2\}, \ldots, \{x_{n-2}\}$, and $\{y_1, y_2\}$. Every $\mathcal{C}$-edge has to involve vertices with a common color by φ, moreover the color class $\{y_1, y_2\}$ cannot be a $\mathcal{B}_1$-edge, therefore:

$$(\beta_1) \qquad \{y_1, y_2\} \subseteq C^* \qquad \text{and} \qquad \{y_1, y_2\} \notin \mathcal{B}_1.$$

Taking the union of any two color classes from φ, the obtained $(n-2)$-coloring is not feasible, what can be caused only by breaking some bound b_i. We are going to analyze the various vertex partitions with $n-2$ classes.

- The contraction of any two singletons $\{x_j\}$ and $\{x_k\}$ is forbidden, hence there exists an edge $E_i \supseteq \{x_j, x_k\}$ with bound $b_i = 1$. That is, $\{x_j, x_k\} \in \mathcal{B}_1$ for every $1 \leq j < k \leq n-2$, so that (γ) holds.

- The contraction of any singleton $\{x_j\}$ and $\{y_1, y_2\}$ is also forbidden by some bound b_i, consequently at least one of the alternatives from (δ) holds.

- Any $(n-2)$-partition containing the two non-singleton color classes $\{x_j, y_1\}$ and $\{x_k, y_2\}$ is non-feasible, hence either a $\mathcal{C}$-edge omits both x_j and x_k (since it includes both y_1 and y_2), or there occur $\mathcal{B}_1$ edges in both sets $\{\{x_j, y_1\}, \{x_k, y_2\}\}$ and $\{\{x_j, y_2\}, \{x_k, y_1\}\}$. This means, the implication of (ϵ) is valid.

By assumption, φ is the unique coloring of $\mathcal{H}$; thus, the coloring with singletons and the only two-element color-class $\{x_j, y_k\}$ is non-feasible for all $1 \leq j \leq n-2$ and $1 \leq k \leq 2$. If x_j belongs to each $\mathcal{C}$-edge, the bounds t_i are fulfilled, hence in this case there surely occurs $\{x_j, y_k\}$ as a $\mathcal{B}_1$-edge:

(β_2) If $x_j \in C^*$, then $\{x_j, y_1\} \in \mathcal{B}_1$ and $\{x_j, y_2\} \in \mathcal{B}_1$ hold.

The properties (β_1), (β_2) and (γ) together ensure the existence of a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge on the intersection of $\mathcal{C}$-edges, implying that (β) is fulfilled, too.

Now, assume a hypergraph $\mathcal{H}$ satisfying the conditions $(\alpha) - (\epsilon)$ of the theorem. Unique $(n-1)$-colorability is verified as follows:

- By the requirement (α), the hypergraph admits no n-coloring.

- Consider the $(n-1)$-coloring φ, where the only monochromatic vertex pair is $\{y_1, y_2\}$. According to (β), both y_1 and y_2 are contained in each $\mathcal{C}$-edge, and hence, due to (α) all the bounds from $\boldsymbol{t}$ are satisfied. Since $\{y_1, y_2\} \notin \mathcal{B}_1$, every hyperedge E_i containing both y_1 and y_2, has bound $b_i \geq 2$, whilst each of the remaining hyperedges involves no monochromatic vertex pair. Therefore, all bounds from $\boldsymbol{b}$ are fulfilled, the color partition $\{x_1\}, \{x_2\}, \ldots, \{x_{n-2}\}, \{y_1, y_2\}$ is feasible.

- According to (γ):

  ($\star$) There is no feasible partition with a color class containing both x_j and x_k (for all $1 \leq j < k \leq n-2$).

  Thus, the only possibility for a second $(n-1)$-coloring would be a partition with 2-element color class $\{x_j, y_k\}$ (for some $1 \leq j \leq n-2$ and $1 \leq k \leq 2$). But if $x_j \in C^*$, there is contained a forbidden $\mathcal{B}_1$-edge due to (β), whilst if $x_j \notin C^*$, then some forbidden polychromatic $\mathcal{C}$-edge would arise. Consequently, no $(n-1)$-coloring different from φ can be feasible.

- To prove that no $(n-2)$-colorings exist:

  ($\star\star$) There is no feasible partition containing $\{x_j, y_1, y_2\}$ as a color class.
  If $x_j \in C^*$, the class contains two forbidden $\mathcal{B}_1$-edges, due to (β). And if $x_j \notin C^*$, the property (δ) ensures that there occurs either a forbidden $\mathcal{B}_1$-edge in the 3-element color class, or this class is involved in a $\mathcal{B}_2$-edge. All these cases are impossible.

  ($\star\star\star$) The pairs $\{x_j, y_1\}$ and $\{x_k, y_2\}$ cannot be color classes simultaneously.
  Such a coloring is trivially non-feasible if there exists a $\mathcal{C}$-edge containing neither x_j nor x_k. Also, if at least one of the vertices x_j and x_k is contained in C^*, the partition is forbidden by a $\mathcal{B}_1$-edge according to (β). In the third case, when all $\mathcal{C}$-edges meet $\{x_j, x_k\}$ but their intersection doesn't, the conditions of (ϵ) are satisfied, hence its conclusion excludes the feasibility of this partition.

  The claims ($\star$), ($\star\star$) and ($\star\star\star$) together imply that the hypergraph $\mathcal{H}$ admits no $(n-2)$-coloring.

- Because of ($\star$), the vertices $x_1, x_2, \ldots, x_{n-2}$ have mutually distinct colors in every feasible coloring, therefore $\mathcal{H}$ admits no coloring with fewer than $n-2$ colors.

Thereupon, the hypergraph is uniquely $(n-1)$-colorable, and this completes the proof. □

There is no restriction for the exact value of bounds $b_i \geq 3$ in the characterization, therefore we immediately get the following corollary:

Corollary 21. *Let $\mathcal{H}$ and $\mathcal{H}'$ be (T, B)-hypergraphs on n vertices, and suppose that $\mathcal{H}'$ can be obtained from $\mathcal{H}$ by replacing each bound $b_i \geq 3$ with some bound $3 \leq b'_i \leq |E_i|$. Then $\mathcal{H}$ is uniquely $(n-1)$-colorable if and only if so is $\mathcal{H}'$. In particular, concerning unique $(n-1)$-colorability, $\mathcal{H}$ can be reduced to a T-hypergraph supplemented with some $\mathcal{B}_1$- edges and 3-element $\mathcal{B}_2$-edges, that is, with a classical ($\mathcal{D}$-) hypergraph of rank at most three.*

Except for the first property (α), the above characterization gives conditions only for the color-bound function $\boldsymbol{b}$ and for the edges having bound $t_i = |E_i| - 1$. Since the restriction $\max_{E_i \in \mathcal{E}} (|E_i| - t_i) = 1$ can be equivalently expressed with the bound $\boldsymbol{a}$, we get an analogous characterization for uniquely $(n-1)$-colorable (A, B)-hypergraphs, too. The terms $\mathcal{B}_1$ and $\mathcal{B}_2$ are used as above; $\mathcal{C}$-edge means a hyperedge E_i with bound $a_i = 2$.

Theorem 26. *An (A, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{a}, \boldsymbol{b})$ with $|X| = n$ vertices is uniquely $(n-1)$-colorable if and only if the following conditions hold:*

(α') $\max_{E_i \in \mathcal{E}} a_i = 2$.

(β) *The set $C^* := \bigcap_{C \in \mathcal{C}} C$ contains at least two vertices and induces a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge.*

Moreover, denoting by y_1 and y_2 the vertices of the omitted $\mathcal{B}_1$-edge,

(γ) *$X \setminus \{y_1, y_2\}$ is a complete $\mathcal{B}_1$-graph.*

(δ) *For each vertex $x \in X \setminus C^*$, at least one of the relations $\{x, y_1\} \in \mathcal{B}_1$, $\{x, y_2\} \in \mathcal{B}_1$ and $\{x, y_1, y_2\} \subseteq E_i \in \mathcal{B}_2$ holds.*

(ϵ) *For each pair of vertices $x_j, x_k \in X \setminus C^*$, if $\{x_j, x_k\}$ intersects every $\mathcal{C}$-edge, then either there exist $\mathcal{B}_1$-edges $\{z, x_j\}$ and $\{z, x_k\}$ for some $z \in \{y_1, y_2\}$, or there exist $\mathcal{B}_1$-edges $\{z, y_1\}$ and $\{z, y_2\}$ for some $z \in \{x_j, x_k\}$.*

Proof If the condition (α') is valid for a given (A, B)-hypergraph, we can replace each bound $a_i = 2$ by $t_i = |E_i| - 1$, whilst the non-restricting $a_i = 1$ can be rewritten as $t_i = |E_i|$, and we get a chromatically equivalent (T, B)-hypergraph on the same vertex set. The obtained (T, B)-hypergraph is uniquely $(n-1)$-colorable if and only if so is the original (A, B)-hypergraph. Also, the $\mathcal{B}_1$-, $\mathcal{B}_2$- and $\mathcal{C}$-edges are the same, hence in this case the conditions (α')–(ϵ) give an exact characterization for the (A, B)-hypergraph.

On the other hand, if the condition (α') does not hold, then either $\overline{\chi} < n-1$ or $\overline{\chi} = n$ or the hypergraph is uncolorable, so it is not uniquely $(n-1)$-colorable in either case. Hence, (α') is indeed necessary for unique $(n-1)$-colorability. □

As it was our purpose, the characterization theorems make it possible to design polynomial-time algorithms for testing unique $(n-1)$-colorability in (T, B)- and (A, B)-hypergraphs.

Remark 19. *The decision problem wether a given (T, B)- or (A, B)-hypergraph is uniquely $(n-1)$-colorable can be solved in time $O(n^2 m)$, where n and m denote the number of vertices and hyperedges, respectively.*

As a matter of fact, on the one hand it is obvious that the condition $a_i \leq 2$ is necessary for unique $(n-1)$-colorability in *every* stably bounded hypergraph; while, on the other hand, the proof of Theorem 26 shows that if this condition holds, then the color-bound function $\boldsymbol{a}$ can completely be replaced with a suitably chosen $\boldsymbol{t}$. Thus, the following more general result is obtained.

Theorem 27. *The decision problem* UNIQUE $(n-1)$-COLORABILITY *can be solved in polynomial time for (T, A, B)-hypergraphs.*

9 Applications

The results of this Thesis concern mixed, color-bounded and stably bounded hypergraphs. We are going to show that the coloring constraints discussed here can be applied efficiently for modeling problems arising in informatics. In Section 9.1 we give a detailed description of possible applications for the frequency assignment problem. It will be shown that different forms of this problem, which have various kinds of non-classical graph coloring models, can be formulated in a unified and natural way in terms of color- and stably bounded hypergraphs. In Section 9.2 we study the possible applications for dependability problems of complex IT systems, and finally, we discuss shortly some further types of practical problems.

Generally, the main examples concern resource allocation, which is one of the core problems in IT design. This process maps tasks, processes etc. to be carried out onto their physical or logical execution platform. The formalism of mixed and stably bounded hypergraphs fits well to the core notions in resource allocation.

- Coloring is an effective means for denoting resources and/or resource types.
- $\mathcal{C}$-type edges are able to express requirements on resource (or resource type) compatibility, i.e. that two or more tasks have to use identical resources (or resource types).
- $\mathcal{D}$-type edges can be used for the formulation of incompatibilities between tasks.
- Color-bounds introduced into the new notion of stably bounded hypergraph provide an effective way to integrate quantitative requirements.

9.1 Frequency assignment problem

One of the characteristic applications of graph coloring theory is the 'frequency (channel) assignment problem' (FAP) which arises in planning television and radio broadcasting, mobile telephone networks, and satellite communication. FAP arises in these different forms with specific characteristics, hence also the mathematical models involve different constraints. We shall show that color-bounded and stably bounded hypergraphs provide a natural and common frame for modeling each of these variants.

The frequency assignment problem first appeared in the 1960's. Metzger was the first who pointed out that it can be successfully treated using graph coloring and mathematical optimization models [43]. The exceptionally fast development of wireless communication has placed FAP in the center of interest, as it is indicated by the large amount of results surveyed e.g. in the papers [48, 41, 21, 10]. Presently, transition from the current analog technology to the digital one provides new challenging frequency assignment problems.

The basic problem involves assigning frequencies to a given collection of wireless communication connections (transmitters), under constraints which ensure that the interference stays at an acceptably small level. The frequencies assigned to transmitters x_i and x_j incur interference, resulting in quality loss of the signal, if x_i and x_j are geographically close[11] to each other, and the frequencies assigned to them are close (or harmonics) on the electromagnetic band. The problem arises in every case when a new transmitter is settled, and the solution requires the application of combinatorial optimization techniques.

For every form of the problem we describe the corresponding non-classical graph coloring constraint and then give an equivalent model using color- and stably bounded hypergraphs.

In each variant, the transmitters $\{x_1, x_2, \ldots, x_n\}$ are taken to be the vertices of a graph, and an edge $\{x_i, x_j\}$ means that the corresponding transmitters can interfere. The colors (i.e., frequencies or channels) assigned to the vertices have to be chosen from a given range $\{1, 2, \ldots, K\}$, and the obtained assignment f is considered feasible if prescribed coloring constraints are fulfilled, what means that excessive interference is avoided.

In the simplest model the only condition is that any two adjacent vertices have to receive different colors. This results in vertex coloring of the graph in classical sense. But in most of the cases this version seems to be oversimplified as it neglects interference and hence, needs various extensions.

9.1.1 Mobile telephone networks: Distance-labeling

In a more accurate model, we distinguish between 'close' and 'very close' transmitters, and different constraints can be prescribed for these two cases. The corresponding non-classical graph coloring is called '$L(d,1)$-labeling' ($d \geq 1$). The coloring constraint is that any two adjacent ('very close') vertices x_i and x_j should receive frequencies with at least d apart: $|f(x_i) - f(x_j)| \geq d$, whilst any two vertices having a common neighbor (at distance two in the graph) should get different colors.

'Distance-labeling' is a generalized version of this model, in which we can give conditions not only for the first and second neighborhoods, but also for the third, fourth, ..., k-th ones with thresholds $d_1 \geq d_2 \geq \ldots \geq d_k$. That is, if the vertices x_i and x_j are at distance $\ell \leq k$ apart in the graph[12], then for the received colors (frequencies) $|f(x_i) - f(x_j)| \geq d_\ell$ must be fulfilled [26].

[11] More precisely, this property is determined not only by the distance of x_i and x_j but it is also influenced by the directions of the transmissions, the relief of the environment and weather conditions.

[12] The distance of two vertices in a graph is defined as the number of edges in a shortest path connecting them.

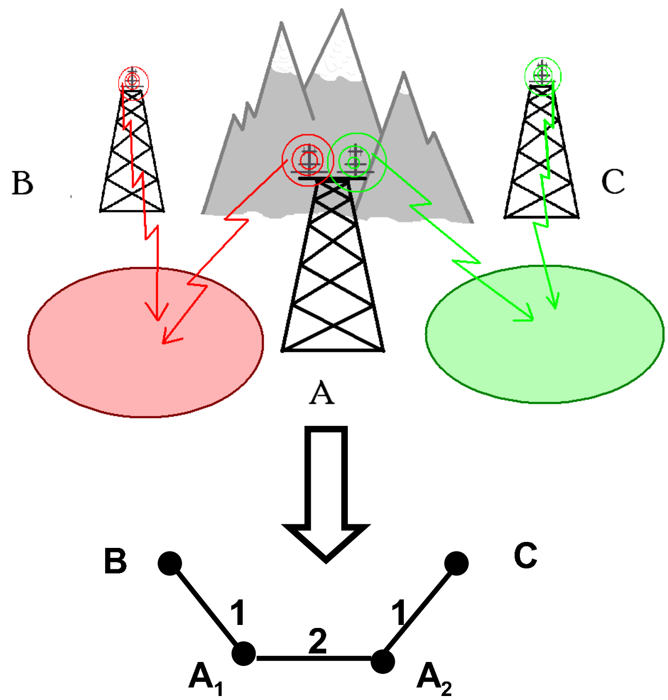

Figure 10: Three sites A, B, C with two antennae on the first one, and the corresponding graph with channel separation constraints on the edges

From the practical side, this approach is motivated by planning mobile telephone networks. In these systems an antenna transmits signals in a certain sector (cell), with a certain number of transmitters depending on the density of cells. Under low-traffic conditions the number is small (e.g., 3) but in metropolitan areas it may well be above 10. In order to avoid channel interference, separation constraints are put. The usual requirements are: separation at least 3 or 2 depending on whether the two transmitters in question belong to the same antenna, or to different antennae of the same site, respectively. Moreover, if two transmitters of different sites have cells with nonempty overlap, their frequencies must differ by at least 1. A simplified illustrative example is shown in Figure 10.

9.1.2 Mobile telephone networks: Constraint matrix

The most precise model concerning frequency assignment in mobile telephone networks is a generalization of the previous ones. Introducing the notion of 'constraint matrix', the required minimum difference of frequencies can be independently specified for each edge. If the transmitters x_i and x_j together may cause interference, the edge $\{x_i, x_j\}$ is associated with a threshold $\ell_{i,j}$, which means the minimum channel separation to avoid interference. That is, the inequality $|f(x_i) - f(x_j)| \geq \ell_{i,j}$ must be satisfied [41, 42]. Here we can take into account different (e.g., geographical) conditions which can restrict pairs of overlapping cells in different ways.

This version — and correspondingly, also the more particular previous ones — can be expressed in terms of color-bounded hypergraphs. Let the vertex set be the union of the set $X = \{x_1, x_2, \dots, x_n\}$ of transmitters and the set $C = \{c_1, c_2, \dots, c_K\}$ of available frequencies. To distinguish frequencies from each other, we take the set C as a hyperedge with bounds $(\boldsymbol{s}(C), \boldsymbol{t}(C)) = (K, K)$ and then, assuring that there will be assigned frequencies only corresponding to the elements of C, we take the hyperedge $X \cup C$ also with bounds (K, K). For every pair $\{x_i, x_j\}$ of transmitters which can cause interference, we consider the $(\ell_{i,j}+2)$-element hyperedge $\{x_i, x_j\} \cup \{c_m, c_{m+1}, \dots, c_{m+\ell_{i,j}-1}\}$ with bounds $(\ell_{i,j}+1, \ell_{i,j}+2)$ for every $1 \le m \le K-\ell_{i,j}+1$. The lower color-bound $\ell_{i,j}+1$ forces that at most one of the vertices x_i, x_j can receive its color (i.e., frequency) from the considered $\ell_{i,j}$ consecutive elements of C. That is, in a proper coloring, the colors assigned to any pair $\{x_i, x_j\}$ of adjacent transmitters fulfill the inequality $|f(x_i) - f(x_j)| \ge \ell_{i,j}$. Consequently, the proper colorings of this color-bounded hypergraph are in one-to-one correspondence with the frequency assignments satisfying the conditions given in the constraint matrix.

9.1.3 Television and radio broadcasting

When the frequency assignment problem concerns television and radio broadcasting, the constraints partially differ from the previous case. The cause lies mainly in the used frequency band itself, which includes higher harmonics of the frequencies. Because of the presence of harmonics, it is more appropriate to prescribe the set T of forbidden differences between frequencies than imposing thresholds on them. For example, the information has been revealed that for UHF TV broadcasting the set of disallowed differences is $T = \{0, 1, 2, 5, 14\}$.

In the corresponding graph coloring model any two adjacent vertices x_i and x_j must receive channels whose separation is not in the set T; that is, $|f(x_i)-f(x_j)| \notin T$. Moreover, in generalized T-coloring the sets $T_{i,j}$ of forbidden separations can be specified for each edge $\{x_i, x_j\}$. A coloring f is considered proper if, for every edge $\{x_i, x_j\}$, the difference between the frequencies assigned to its vertices is not forbidden [27, 48, 49, 17].

Modeling this problem with color-bounded hypergraphs, consider the vertex set $\{x_1, x_2, \dots, x_n\} \cup \{c_1, c_2, \dots, c_K\}$ where the vertices x_i and c_ℓ correspond to the transmitters and frequencies, respectively, as in the previous description. Similarly, we create hyperedges C and $X \cup C$ with bounds (K, K). For every pair x_i, x_j of transmitters which can cause interference, we take the edge $\{x_i, x_j\}$ with bounds $(2, 2)$, moreover consider every forbidden color-pair (p, q) for which $|p-q| \in T_{i,j}$, and create the hyperedge $\{x_i, x_j, c_p, c_q\}$ with bounds $(3, 4)$. The lower color-bound 3 forces that there can occur no two transmitters with forbidden frequency-pairs.

9.1.4 When only a subset of frequencies is available

In many cases, for an individual transmitter only a specifically restricted subset of frequencies is available. For example, if a transmitter of a mobile telephone network is close to the border of a country, division rules often lead to a substantial reduction in channel availability. This situation can be expressed by the list-coloring versions of the above colorings [52, 37].

List coloring means an input graph $G = (X, E)$ with a set L_i of allowed colors for each vertex $x_i \in X$; the question is whether the lists admit a proper coloring, that is a vertex coloring f such that $f(x_i) \in L_i$ for all vertices $x_i \in X$ and $f(x_i) \neq f(x_j)$ for all edges $\{x_i, x_j\} \in E$.

It was first proved in [37] that the list coloring problem on graphs can be expressed in terms of the colorability problem of a suitably constructed mixed hypergraph. The reduction has a fairly transparent structure: the vertex set of the mixed hypergraph is $X \cup \left(\bigcup_{x_i \in X} L_i\right)$, the $\mathcal{D}$-edges are the edges of G and the pairs inside $\bigcup_{x_i \in X} L_i$, and the $\mathcal{C}$-edges are the sets $\{x_i\} \cup L_i$. This mixed hypergraph is colorable if and only if the original graph G admits a list coloring.

This theoretical result exactly shows the way how we can extend the above hypergraph models of frequency assignment problems to their list coloring versions.

9.1.5 Multiple interference

Finally, we mention a variant of interference which is ignored in the standard approach. There can occur more than two transmitters using close frequencies, and these multiple signals may disturb the quality of communication, even if the pairwise combinations of frequencies are not forbidden. There exist only very few models taking into consideration this multiple interference [10]. Clearly, using hypergraphs we can model not only binary relations; hence, in our new model also those cases can be treated efficiently where combinations of frequencies for more than two transmitters are forbidden.

9.2 Resource allocation and dependability

Dependability in IT is the cover notion of techniques assuring that the user can rely on the services.

The typical means of fault-tolerance aiming at a proper (fault-free) operation even in the case of the presence of faults is the introduction of some form of redundancy. If a resource fails, all the functionalities supported by it are taken over by some backup resource after failover [14].

- $\mathcal{C}$-type conditions can easily express the demand for identical resources (as in modular replication), in which case the primary and all of the backup resources are of the same type. A typical example is database replication creating multiple copies of an important database. If any of them becomes corrupted, another one can be used by the application.

The lower bound in coloring expresses here the minimum number of replicas needed for the assurance of a given level of dependability (obviously, the higher number of replicas, the more dependable is the system), while the upper bound may refer to the cost limit allowed.

- $\mathcal{D}$-type conditions express diversity. In the context of dependability, diversity is the main means protecting systems against so-called common-mode faults, i.e. single faults corrupting multiple instances.

  For example, while database replication protects the application against uncorrelated faults (e.g. a disk fails), all of the replicas may simultaneously fail if the database software used in all the replicas has a bug.

  Design or deployment for diversity is the principle for the avoidance of multiple correlated failures originating in a single fault as root cause. Typical example is n-version programming in which the same application is implemented by different teams using different languages, operating systems and development technologies.

  The bounds here can be used once again for expressing the required level of diversity (and correspondingly, dependability), while majorizing costs.

For instance, Information Technology has become a mission-critical component in the operation of businesses, as more and more business services and business support services depend on it. Therefore, finding faulty components in the infrastructure and bounding their impact on the services provided have high priority.

As mentioned before, mixed and stably bounded hypergraphs can be used as models for the allocation of services to resources. We can specify this problem regarding the field of service-oriented architecture (SOA) and security [46]. An optimal service configuration is synthesized using constraints which represent non-functional requirements. For instance, services which must be deployed into the same authorization and communication domain for security reasons or must belong to the same trust circle will form a $\mathcal{C}$-edge. As an example, a typical requirement in e-commerce SOA process is that payment-related subservices have to be executed strictly inside the bank accomplishing it.

Groups with the need for multiple authorities / trust domains will be considered as $\mathcal{D}$-edges. Authentication (that is, the check of the user's identity) is a typical problem in e-Business processes. Here the usual role is that requester and checker components have to be different (no self-identification allowed) in order to avoid fake authentication after the corruption of an application system.

The above constraints can be supplemented with conditions on dependability and fault-tolerance. In many cases there are requirements on the number of available service instances; that is, multiple instances of services have to be deployed on machines (or functionalities/responsibilities assigned to services). These constraints can be expressed efficiently using the technique that will be described in the next section for cases where the number of fixed types is prescribed.

9.3 Some further applications

9.3.1 Scheduling of file transfers

Consider a given set of file transfers, where each transfer needs the simultaneous work of two prescribed processors, and assume that every task takes one unit of time. Moreover, every processor p_i has an upper bound b_i which means that it can participate in at most b_i transfers at the same time. The goal is to find a scheduling which respects the bounds b_i and requires minimum time. The problem can naturally be modeled as an edge coloring problem [40], where the vertices correspond to the processors, an edge means a prescribed file transfer, and the color assigned to the edge stands for the time slot when the transfer will be done. The bound b_i is interpreted in the coloring such that there can be at most b_i edges of the same color which are incident to vertex p_i. Hence, the original problem has a concise and natural modeling with stably bounded edge coloring of a graph, which corresponds (as it is discussed in Chapter 6) to a vertex coloring of a stably bounded hypergraph.

9.3.2 Data access in parallel memory

Hypergraph strong coloring [15] means that each hyperedge has to be colored in polychromatic way; i.e., any two vertices in the same hyperedge have to get different colors, and the minimum possible number of colors is of interest. This problem arises, for example, when we require conflict-free access of data in parallel memory modules. This can also be modeled in terms of color-bounded hypergraphs. Data elements are considered as vertices, whilst every hyperedge contains data elements which should be stored in different memory modules to be processed in parallel. Clearly, if each hyperedge E_i is associated with bounds $s_i = t_i = |E_i|$ then the color-bounded hypergraph obtained models exactly the described problem. This simple case can be expressed in terms of classical graph coloring, too, but the hypergraph representation uses fewer constraints and hence is more economical.

The more general problem originates from applications where the numbers of available colors (i.e., memory modules) are usually bounded by a fixed number k, which can be smaller than the maximum edge size in the hypergraph. In this situation the strong coloring constraint has to be relaxed, but we can prescribe that inside each hyperedge the k colors have to be spread nearly equally. The main goal is to minimize the cardinality of largest monochromatic subset inside each hyperedge [30]. The theory of stably bounded hypergraphs offers an adequate frame to describe this problem. Every hyperedge E_i has to be associated with bounds $s_i = t_i = \min(|E_i|, k)$ and $a_i = b_i = \left\lceil \frac{|E_i|}{k} \right\rceil$, moreover we create a hyperedge F, containing all the vertices, with bounds $\boldsymbol{s}(F) = \boldsymbol{t}(F) = k$. A proper coloring clearly yields minimum value for the maximum cardinality of monochromatic subsets inside the edges, and that exactly k colors are used.

Moreover, if we want to require further that inside each hyperedge E_i, every color should appear either $\left\lceil \frac{|E_i|}{k} \right\rceil$ or $\left\lfloor \frac{|E_i|}{k} \right\rfloor$ times, this can be done by creating disjoint sets E_i' of $\left\lceil \frac{|E_i|}{k} \right\rceil k - |E_i|$ new vertices each, and by supplementing the E_i with them. Each new vertex will be contained in precisely three hyperedges: in a supplemented $E_i \cup E_i'$ with unchanged bounds $a_i = b_i = \left\lceil \frac{|E_i|}{k} \right\rceil$, in a big new edge $F' = \bigcup_{E_i \in \mathcal{E}} (E_i \cup E_i')$ of all vertices with bounds $\boldsymbol{s}(F') = \boldsymbol{t}(F') = k$, and in the new edge E_i' containing only the new vertices created for E_i with bounds $\boldsymbol{a}(E_i') = \boldsymbol{b}(E_i') = 1$. A proper coloring of this stably bounded hypergraph means that the k colors occur equally in any edge, and this yields the possible most equitable spread of the k colors on the original edges, too. That is, the conflict minimization problem can be modeled by colorings of the stably bounded hypergraph obtained.

9.3.3 Prescribing the number of occurrences for fixed types

A vertex coloring means a partition of the vertex set satisfying given rules. In some practical cases the distinguished types (colors) are given in advance, and it is prescribed for a group that precisely m_i elements of it should belong to type i. (A more precise notation is $m_{i,j}$, where j means the index of the group considered.) We can equivalently formulate this restriction using stably bounded hypergraphs.

Let ℓ denote the maximum cardinality of the groups on which the $m_{i,j}$ conditions have been described. We consider the types (colors) $C = \{1, 2, \ldots, K\}$ given beforehand as additional vertices of the hypergraph. To assure that the above colors are pairwise distinct, we create the hyperedge C with bounds $\boldsymbol{s}(C) = \boldsymbol{t}(C) = K$. Then we create ℓ copies for each color i, join them with vertex i by an edge C_i, and assure their monochromaticity with bound $\boldsymbol{t}(C_i) = 1$. After these arrangements the essential constraint can be expressed easily: If a vertex subset E_j should contain precisely $m_{i,j}$ vertices of type i, then we take E_j and the ℓ copies of color i, and join them with a hyperedge $E_{i,j}'$ having bounds $\boldsymbol{a}(E_{i,j}') = \boldsymbol{b}(E_{i,j}') = \ell + m_{i,j}$. Since $|E_j| \leq \ell$ holds by the choice of ℓ, each new hyperedge $E_{i,j}'$ will impose a condition exactly on type i. We note that for one group we can take more restrictions concerning occurrences of different types, and the conditions can also be given in forms 'at least (or at most) $m_{i,j}$ vertices should be assigned to type i'.

These constraints appear in designing fault-tolerant complex systems, when some prescribed critical components have to be duplicated or triplicated. The above model is useful when these conditions have to be satisfied together with further partition constraints on groups of components.

9.3.4 Applications from the earlier literature

Chapter 12 of the monograph [59] is devoted to some applications of mixed hypergraphs, describing examples from the fields of informatics, molecular biology and genetics of populations. Also in [33] several practical and theoretical applications of mixed hypergraphs are discussed. Concerning the application of S-hypergraphs there can be found examples from economy in the paper [19].

Due to its strength, our new model will probably lead to further theoretical and practical applications.

10 Summary

In this work we discussed set partitions under several types of constraints. The first part was devoted to our new results on mixed hypergraph coloring. In this model two types of local conditions are used, and this complex structure makes it possible to handle extremal and existence problems arising in different fields. They can serve as models for a large variety of applied problems, e.g. in molecular biology, in sociology, in informatics, and especially in mobile communication for frequency assignment problems.

We proved here a ten-year-old conjecture concerning the characterization of $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees. Due to the constructive method of proof, also a polynomial-time algorithm that finds a coloring with maximum number of colors has been obtained.

Another long-standing open problem was to determine the minimum number of hyperedges in an r-uniform $\mathcal{C}$-hypergraph that has only trivial colorings; i.e., it cannot be colored with more than $r-1$ colors. We have proved an asymptotically tight estimate for this minimum number. Generalization of this problem leads to many new exciting questions and establishes connections among several intensively studied classical parts of discrete mathematics.

The study of possible numbers of colors in the colorings of r-uniform mixed hypergraphs indicates that coloring properties do not change considerably if we fix the cardinality of hyperedges.

A previous expectation, regarding efficient recognizability of mixed hypergraphs having 'uniquely colorable' vertex order, has been refuted. We have proved that this problem is NP-complete, even if the input is restricted to uniquely colorable hypergraphs.

Color-bounded and stably bounded hypergraphs had not been considered before, they were introduced in our publications. If we put lower and upper bounds on the maximum cardinality of polychromatic and monochromatic subsets of hyperedges, on the one hand many problems can be modeled with simpler structure and in a more natural way than in the case of mixed hypergraphs, and on the other hand there are problems that cannot be described in the earlier model.

In the new structure classes we have studied basic properties; e.g., time complexity of testing colorability, feasible sets of interval hypergraphs and hypertrees, moreover hypergraphs with other restricted structures have also been considered. For some particular types, polynomial-time algorithms have been designed.

We would like to highlight two results obtained in this second part. It is quite a surprise, how central role the color-bounded hypertrees play in this theory. In contrast to their strongly restricted structure, they can model a wide range of colorability problems. They represent nearly all color-bounded hypergraphs, regarding not only feasible sets but also chromatic polynomials. Another interesting result concerns the comparison of different subclasses derived from stably bounded hypergraphs. The model (S, A) is universal, but the hierarchy of (S, T) and (A, B) is not stable, it depends on the type of question considered. It looks a challenging 'meta-problem' to determine the characteristics of problems that can be interpreted more appropriately in one model than in the other.

Finally, we emphasize the results on chromatic polynomials. Our characterization for polynomials of non-1-colorable structures is valid for mixed, color-bounded, stably bounded and pattern hypergraphs as well. Furthermore, the hierarchy of the sets of possible chromatic polynomials is established among subclasses of mixed and stably bounded models.

We expect that continuing the study of mixed, color-bounded and stably bounded hypergraphs, further important results can be obtained, including the development of new efficient algorithms.

In research, not only solving earlier problems but also asking new questions is important. We can anticipate, this area — set partitions under local constraints — and its practical applications will offer many interesting new questions and directions in the future.

List of contributions

1.1. *(Corollary 3) A $\mathcal{C}$-hypertree is $\mathcal{C}$-perfect if and only if it contains no monostar as an induced subhypergraph. Moreover, $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees can be $\bar{\chi}$-colored in polynomial time.*

1.2. *(Theorems 4 and 5) The following decision problems are* NP*-complete on the class of $\mathcal{C}$-hypertrees:*

- *Does the hypertree $\mathcal{T}$ contain an induced monostar?*
- *Is the hypertree $\mathcal{T}$ colorable with $\alpha_C(\mathcal{T})$ colors?*

1.3. *(Theorem 6) Over the class of $\mathcal{C}$-hypertrees there exists a polynomial-time algorithm whose output is either an induced monostar subhypergraph or a proper coloring of $\mathcal{T}$ with $\alpha_C(\mathcal{T}) = \bar{\chi}(\mathcal{T})$ colors.*

1.4. *(Theorem 7) Given a uniquely colorable mixed hypergraph $\mathcal{H}$ with its coloring as input, it is* NP*-complete to decide whether $\mathcal{H}$ has a UC-ordering.*

2.1. *(Theorem 1) Let $r \geq 3$ be an integer, and S a non-empty finite set of positive integers. There exists an r-uniform mixed hypergraph $\mathcal{H}$ with at least one hyperedge and having feasible set $\Phi(\mathcal{H}) = S$ if and only if*

(i) $\min(S) \geq 2$ *and S contains all integers between* $\min(S)$ *and $r-1$ (this means restriction only in the case of* $\min(S) < r-1$*), or*

(ii) $\min(S) = 1$ *and S is of the form of $S = \{1, \ldots, \bar{\chi}\}$ for some natural number $\bar{\chi} \geq r-1$.*

Moreover, S is the feasible set of some r-uniform bi-hypergraph with $\mathcal{C} = \mathcal{D} \neq \emptyset$ if and only if it is of type (i).

2.2. *(Theorem 2) For the minimum number $f(n,r)$ of hyperedges in an r-uniform $\mathcal{C}$-hypergraph with upper chromatic number $r-1$ the following estimates hold for all integers $n > r > 2$:*

(i) $f(n,r) \leq \frac{2}{n-1}\binom{n-1}{r} + \frac{n-1}{r-1}\left(\binom{n-2}{r-2} - \binom{n-r-1}{r-2}\right)$ *for all n and r.*

(ii) $f(n,r) = (1+o(1))\,\frac{2}{r}\binom{n-2}{r-1}$ *for all $r = o(n^{1/3})$ as $n \to \infty$.*

3.1. *(Theorems 14 and 15) Every colorable color-bounded interval hypergraph and Rooted Directed Path hypergraph has a gap-free chromatic spectrum, and its lower chromatic number is equal to* s.

3.2. *(Theorems 16 and 17) Let* S *be a finite set of positive integers. There exists a color-bounded hypertree* $\mathcal{T}$ *with feasible set* $\Phi(\mathcal{T}) = S$ *if and only if*

(*i*) $\min(S) = 1$ *or* $\min(S) = 2$, *and* S *contains all integers between* $\min(S)$ *and* $\max(S)$, *or*

(*ii*) $\min(S) \geq 3$.

4.1. *(Theorems 10, 20 and 21)*

- *If a* (T, A, B)-, (T, B)- *or* (A, B)-*hypergraph has a gap of size* $k \geq 1$ *in its feasible set, then it has at least* $2k + 4$ *vertices.*
- *If an* (S, A)-, (S, T)-, *or stably bounded hypergraph has a gap of size* $k \geq 1$ *in its feasible set, then it has at least* $k + 5$ *vertices.*

Moreover, all the above bounds are sharp.

4.2. *(Theorems 18 and 23) The decision problem of colorability is* NP-*complete on each of the following classes of hypergraphs:*

- *3-uniform* (S, T)-*hypertrees,*
- *3-uniform* (S, A)-*hypertrees,*
- *3-uniform* (T, B)-*hypertrees,*
- *3-uniform* (A, B)-*hypertrees.*

4.3. *(Theorems 12, 24 and 25)*

- *The decision problem of unique* $(n-1)$-*colorability is* co-NP-*complete for (S,A)-, (S,T)-, and stably bounded hypergraphs.*
- *The decision problem of unique* $(n-1)$-*colorability can be solved in polynomial time for (T,A,B)-, (T,B)- and (A,B)-hypergraphs.*

5. *(Theorem 9) Let* $P(\lambda) = \sum_{k=0}^{\ell} a_k \lambda^k \not\equiv 0$ *be a polynomial such that* $P(1) = 0$*, i.e.* $\sum_{k=0}^{\ell} a_k = 0$*. The following properties are equivalent.*

1. $P(\lambda)$ *is the chromatic polynomial of a stably bounded hypergraph.*
2. $P(\lambda)$ *is the chromatic polynomial of a color-bounded hypergraph.*
3. $P(\lambda)$ *is the chromatic polynomial of a mixed hypergraph.*
4. $P(\lambda)$ *satisfies all of the following conditions.*
   - *(i) All coefficients* a_k *of* $P(\lambda)$ *are integers.*
   - *(ii) The leading coefficient* a_ℓ *is positive.*
   - *(iii) The constant term* a_0 *is zero.*
   - *(iv) For every positive integer* $j \leq \ell$*, the inequality*

$$\sum_{k=j}^{\ell} a_k \cdot S(k,j) \geq 0$$

is valid.

References

I. Publications from the results of the Thesis

[1] Cs. Bujtás and Zs. Tuza: Mixed colorings of hypergraphs. *Electronic Notes in Discrete Mathematics* 24 (2006), 273–275.

[2] Cs. Bujtás and Zs. Tuza: Orderings of uniquely colorable hypergraphs. *Discrete Applied Mathematics* 155 (2007), 1395–1407. (Impact Factor 0.577)

[3] Cs. Bujtás and Zs. Tuza: Uniform mixed hypergraphs: The possible numbers of colors. *Graphs and Combinatorics*, 24 (2008), 1–12. (Impact Factor 0.175)

[4] Cs. Bujtás and Zs. Tuza: Color-bounded hypergraphs, I: General results. *Discrete Mathematics*, in print. (Impact Factor 0.347)

[5] Cs. Bujtás and Zs. Tuza: Color-bounded hypergraphs, II: Interval hypergraphs and hypertrees. Manuscript, 2006, submitted. (Accepted by guest editors of Discrete Mathematics, expected Impact Factor 0.347)

[6] Cs. Bujtás and Zs. Tuza: Color-bounded hypergraphs, III: Model comparison. *Applicable Analysis and Discrete Mathematics* 1 (2007), 36–55.

[7] Cs. Bujtás and Zs. Tuza: Voloshin's conjecture for $\mathcal{C}$-perfect hypertrees. Manuscript, 2007, submitted.

[8] Cs. Bujtás and Zs. Tuza: Smallest set-transversals of k-partitions. Manuscript, 2007.

[9] Cs. Bujtás and Zs. Tuza: Partition-crossing hypergraphs. Manuscript, 2007.

II. Further references

[10] K. I. Aardal, C. P. M. van Hoesel, A. M. C. A. Koster, C. Mannino and A. Sassano: Models and solution techniques for the frequency assignment problem. *4OR* 1(4) (2003), 261–317.

[11] N. Ahuja and A. Srivastav: On constrained hypergraph coloring and scheduling. In: Approximation Algorithms for Combinatorial Optimization, Springer-Verlag, *Lecture Notes in Computer Science* 2462 (2002), 14–25.

[12] R. Ahlswede: Coloring hypergraphs: a new approach to multi-user source coding, Parts I and II. *Journal of Combinatorics, Information and System Sciences* 4 (1979), 76–115 and 5 (1980), 220–268.

[13] G. Bacsó, Zs. Tuza and V. I. Voloshin: Unique colorings of bi-hypergraphs. *Australasian Journal of Combinatorics* 27 (2003), 33–45.

[14] A. Balogh, A. Pataricza and J. Rácz: Scheduling of embedded time-triggered systems. *EFTS '07: Proceedings of the 2007 workshop on Engineering fault tolerant systems.* ACM, 2007.

[15] C. Berge: *Graphs and Hypergraphs*. Dunod, Paris, 1970, and North-Holland, New York, 1973.

[16] E. Bulgaru and V. Voloshin: Mixed interval hypergraphs. *Discrete Applied Mathematics* 77 (1997), 24–41.

[17] M. B. Cozzens and F. S. Roberts: T-Colorings of graphs and the channel assignment problem. *Congressus Numerantium* 35 (1982), 191–208.

[18] K. Dohmen: A broken-circuits-theorem for hypergraphs. *Archives of Mathematics* 64 (1995), 159–162.

[19] E. Drgas-Burchardt and E. Łazuka: On chromatic polynomials of hypergraphs. *Applied Mathematics Letters* 20 (2007), 1250–1254.

[20] Z. Dvořák, J. Kára, D. Král' and O. Pangrác: Pattern hypergraphs. Manuscript, 2004.

[21] A. Eisenblätter, M. Grötschel and A. M. C. A. Koster: Frequency assignment and ramifications of coloring. *Discussiones Mathematicae Graph Theory* 22 (2002), 51-88.

[22] P. Erdős and A. Hajnal: On chromatic number of graphs and set systems. *Acta Mathematica Academiae Scientiarum Hungaricae* 17 (1966), 61–99.

[23] T. Gallai, unpublished. (Early 1950's.)

[24] A. Gamst and W. Rave: On frequency assignment in mobile automatic telephone systems. *Proceedings of GLOBECOM'82, IEEE*, (1982), 309–315.

[25] M. C. Golumbic: *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, New York, 1980.

[26] J. R. Griggs and R. K. Yeh: Labeling graphs with a condition at distance two. *SIAM Journal on Discrete Mathematics* 5 (1992), 586–595.

[27] W. K. Hale: Frequency assignment: Theory and applications. *Proceedings of the IEEE* 68 (1980), 1497–1514.

[28] T. R. Jensen and B. Toft: *Graph Coloring Problems.* John Wiley & Sons, Inc., New York, 1995.

[29] T. Jiang, D. Mubayi, V. Voloshin, Zs. Tuza and D. West: The chromatic spectrum of mixed hypergraphs. *Graphs and Combinatorics* 18 (2002), 309–318.

[30] D. Kaznachey, A. Jagota and S. Das: Neural network-based heuristic algorithms for hypergraph coloring problems with applications. *Journal of Parallel Distributed Computing* 63 (2003), 786–800.

[31] D. Král': A counter-example to Voloshin's hypergraph co-perfectness conjecture. *Australasian Journal of Combinatorics* 27 (2003), 25–41.

[32] D. Král': On feasible sets of mixed hypergraphs. *The Electronic Journal of Combinatorics* 11 (2004), R19.

[33] D. Král': Mixed hypergraphs and other coloring problems. *Discrete Mathematics* 307 (7–8) (2007), 923–938.

[34] D. Král', J. Kratochvíl, A. Proskurowski and H.-J. Voss: Coloring mixed hypertrees. Proceedings of the 26th Workshop on Graph-Theoretic Concepts in Computer Science, *Lecture Notes in Computer Science* 1928, Springer-Verlag, 2000, 279–289.

[35] D. Král', J. Kratochvíl and H.-J. Voss: Mixed hypercacti. *Discrete Mathematics* 286 (2004), 99–113.

[36] D. Král', J. Kratochvíl and H.-J. Voss: Mixed hypergraphs with bounded degree: edge-coloring of mixed multigraphs. *Theoretical Computer Science* 295 (2003), 263–278.

[37] J. Kratochvíl, Zs. Tuza and M. Voigt: New trends in the theory of graph colorings: Choosability and list coloring. In: Contemporary Trends in Discrete Mathematics (R. L. Graham et al., eds.), *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* 49, AMS, 1999, 183–197.

[38] L. Lovász: Coverings and colorings of hypergraphs. *Congressus Numerantium* 8 (1973), 3–12.

[39] C.-J. Lu: Deterministic hypergraph coloring and its applications. *Proceedings of the 2nd International Workshop on Randomization and Approximation Techniques in Computer Science*, 1998, 35–46.

[40] D. Marx, Thesis. Budapest University of Technology and Economics, 2004.

[41] C. McDiarmid: Frequency-distance constraints with large distances. *Discrete Mathematics* 223 (2000), 227–251.

[42] C. McDiarmid and B. Reed: Channel asignment on graphs of bounded treewidth. *Discrete Mathematics* 273 (2003), 183–192.

[43] B. H. Metzger: Spectrum management technique. *Presentation at 38th National ORSA meeting, Detroit, MI,* 1970.

[44] A. Niculitsa and V. Voloshin: About uniquely colorable mixed hypertrees. *Discussiones Mathematicae Graph Theory* 20 (2000), 81–91.

[45] A. Niculitsa and H.-J. Voss: A characterization of uniquely colorable mixed hypergraphs of order n with upper chromatic numbers $n-1$ and $n-2$. *Australasian Journal of Combinatorics* 21 (2000), 167–177.

[46] A. Pataricza, A. Balogh and L. Gönczy: Verification and validation of nonfunctional aspects in enterprise. Enterprise Modeling and Computing with UML (P. Rittgen, ed.), Idea Group Publishing, 2007, 261–303.

[47] K. T. Phelps and V. Rödl: On the algorithmic complexity of coloring simple hypergraphs and Steiner triple systems. *Combinatorica* 4 (1984), 79–88.

[48] F.S. Roberts: New directions in graph theory (with an emphasis on the role of applications). *Annals of Discrete Mathematics* 55 (1993), 13–44.

[49] F.S. Roberts: Some applications of graph theory. In *Combinatorial and Computational Mathematics: Present and Future.* World Scientific Publishing Co. Pte. Ltd, 2001.

[50] A. Srivastav and P. Stangier: Tight approximations for resource constrained scheduling and bin packing. *Discrete Applied Mathematics* 79 (1997), 223–245.

[51] R.E. Tarjan and M. Yannakakis: Simple linear time algorithms to test chordability of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM J. Comput.* 13 (1984), 566–579.

[52] Zs. Tuza: Graph colorings with local constraints — A survey. *Discussiones Mathematicae Graph Theory* 17 (1997), 161–228.

[53] Zs. Tuza and V. Voloshin: Problems and results on colorings of mixed hypergraphs. In *Horizon of Combinatorics*, J. Bolyai Math. Soc., to appear.

[54] Zs. Tuza and V. Voloshin: Uncolorable mixed hypergraphs. *Discrete Applied Mathematics* 99 (2000), 209–227.

[55] Zs. Tuza and V. Voloshin: Linear-time algorithm for the edge-colorability of a graph with prescribed vertex types. *Computer Science Journal of Moldova* 11 (2003), 35-42.

[56] Zs. Tuza, V. Voloshin and H. Zhou: Uniquely colorable mixed hypergraphs. *Discrete Mathematics* 248 (2002), 221–236.

[57] V. Voloshin: The mixed hypergraphs. *Computer Science Journal of Moldova* 1 (1993), 45–52.

[58] V. Voloshin: On the upper chromatic number of a hypergraph. *Australasian Journal of Combinatorics* 11 (1995), 25–45.

[59] V. I. Voloshin: *Coloring Mixed Hypergraphs – Theory, Algorithms and Applications*. Fields Institute Monographs 17, AMS, 2002.

[60] V. Voloshin. Mixed Hypergraph Coloring Web Site: http://spectrum.troy.edu/ voloshin/mh.html.

[61] V. Voloshin and H.-J. Voss: Circular mixed hypergraphs I: Colorability and unique colorability. Proceedings of the Thirty-first Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2000), *Congressus Numerantium* 144 (2000), 207–219.

[62] V. Voloshin and H.-J. Voss: Circular mixed hypergraphs II: The upper chromatic number. *Discrete Applied Mathematics* 154 (2006), 1157–1172.