Algorithms and Structure: Set Partitions Under Local Constraints

PhD Thesis Summary

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2008

1 Motivation and objectives

Graph coloring is a classic model in combinatorial optimization with many modern applications in science and engineering. The rapid development of informatics has created new horizon for these applications.

But not all of the problems, which have occurred, can be directly modeled with classical vertex coloring, where the only local constraint is that any two adjacent vertices have to receive different colors. Hence, problems from real life strongly inspired the appearance and intensive studying of generalized coloring conditions.

Applying the set partition constraints discussed in the Thesis, numerous variants of classical graph coloring can be described in a unified way, providing a model for problems from various fields.

One of the most significant present applications of graph coloring theory is the 'frequency (channel) assignment problem' (FAP) which arises, e.g., in planning television and radio broadcasting, mobile telephone networks and satellite communication.

The basic problem involves assigning frequencies to a given collection of wireless communication connections (transmitters), under constraints which ensure that the interference stays at an acceptably small level. The frequencies assigned to connections x_i and x_j incur interference, resulting in quality loss of the signal, if x_i and x_j are geographically close to each other, and the frequencies assigned to them are close (or harmonics) on the electromagnetic band.

FAP first arose in the 1960's, and Metzger [32] pointed out that it can be efficiently treated using graph coloring and mathematical optimization techniques. The fast development of wireless communication has placed FAP in the center of interest [21, 19, 35, 30, 17, 10]. At the present time, transition from the current analog technology to the digital one provides new challenging frequency assignment problems.

FAP arises in different forms with specific characteristics, hence also the mathematical models involve different constraints.

In each model the transmitters $\{x_1, x_2, \ldots, x_n\}$ are taken to be the vertices of a graph, and an edge $\{x_i, x_j\}$ means that the corresponding transmitters can interfere. The colors (i.e., frequencies or channels), assigned to the vertices, have to be chosen from a set $\{1, 2, \ldots, K\}$, and the obtained assignment f is considered feasible if prescribed coloring constraints are fulfilled, what means that excessive interference is avoided.

- In the basic model the only condition is that any two adjacent vertices have to receive different colors. This results in vertex coloring of the graph in the classical sense. But in most of the cases this version seems to be oversimplified and hence, needs various extensions.

- If we distinguish between 'close' and 'very close' transmitters, different constraints can be prescribed for these two cases. In the model $L(d,1)$-labeling ($d \geq 1$) any two adjacent ('very close') vertices x_i and x_j should receive frequencies with at least d apart: $|f(x_i) - f(x_j)| \geq d$, whilst any two vertices having a common neighbor (in distance two in the graph) should get different colors. 'Distance-labeling' is a generalized version of this model, in which we can give conditions not only for the first and second neighborhoods, but also for the third, fourth, ..., k-th ones with thresholds $d_1, d_2, \ldots, d_k$ [20].

- Introducing the idea of 'constraint matrix', the required minimal difference of frequencies can be independently specified for each edge. The edge $\{x_i, x_j\}$ means that the corresponding transmitters together may cause interference. Moreover, the edge is associated with a threshold $\ell(x_i, x_j)$, which means the minimum channel separation to avoid interference; that is, the inequality $|f(x_i) - f(x_j)| \geq \ell(x_i, x_j)$ must be satisfied [31]. This is the most standard model concerning frequency assignment in mobile telephone networks.

- In certain cases it is more appropriate to prescribe the set T of forbidden differences between frequencies. Adjacent vertices x_i and x_j must receive channels whose separation is not in the set T; that is, $|f(x_i) - f(x_j)| \notin T$. For example, the information has been revealed that for UHF TV broadcasting the set of disallowed differences is $T = \{0, 1, 2, 5, 14\}$. In generalized T-coloring the set of forbidden separations can be specified for each edge [21, 13].

- In many cases only a subset of frequencies are available for transmitters. This situation can be expressed by the list-coloring versions of the above colorings [37, 28].

As it is shown in the dissertation, each coloring constraint described above can be modeled in a natural way applying the new coloring constraints of stably and color-bounded hypergraphs.

There are many further possibilities for applications of the new coloring constraints in informatics. Our main examples belong to the field of resource allocation, as it is discussed in more specified forms concerning the planning of dependable systems and service-oriented architectures (SOA) [34, 12], the scheduling of file transfers, and the optimization of data access [23].

2 Methodology

The theory of mixed hypergraphs is a relatively young but already well-developed field. The first part of the dissertation contains solutions for several important open problems belonging to different parts of this theory, consequently these results are not strongly connected to each other.

In the second part new coloring constraints are introduced. The local conditions prescribe lower and upper bounds for the cardinalities of largest polychromatic and monochromatic subsets on each hyperedge. This model generalizes the notion of mixed hypergraph and other coloring concepts, moreover it provides possibilities for new applications.

As regards methodology, many results are proved in algorithmic or constructive ways, combining these with inductive methods. Structural observations yield characterizations for several subclasses of the set systems considered, and in many cases the characterization naturally corresponds to a polynomial-time recognition algorithm. Moreover, we consider several problems that turn out to be NP-hard, in some cases refuting the previous expectations. The standard way to show that a problem is NP-hard is to reduce it from another NP-hard one. In these reductions we apply time complexity results belonging to different fields.

In the dissertation a powerful algorithmic tool is introduced, which can be applied for color-bounded hypergraphs to obtain another proper coloring (using fewer colors) from a known one. The conditions for its applicability are summarized in the Recoloring Lemma. Beside the possible practical importance, it turns out to be very useful also theoretically. We apply this tool to determine the possible feasible sets and the lower chromatic number for hypergraphs of different structure classes.

Some of the new results, discussed in the Thesis, have quite a technical formulation in their precise and strongest form. Although the introduction of mathematical terms cannot be avoided, in this summary we seek to describe the theorems in a less technical form. Therefore, some statements are inserted here in weaker form than they are asserted and proved in the dissertation, and the direction of possible strengthening is only indicated.

3 Preliminaries — The mixed hypergraph model

Classical vertex coloring of a hypergraph[1], as it was introduced in 1966 by Erdős and Hajnal [18], applies the constraint that each hyperedge should contain at least two vertices with distinct colors. Nearly 30 years later Voloshin introduced the idea of mixed hypergraph [41, 42] in which two types of local coloring constraints are used. The role of $\mathcal{D}$-edges corresponds to the classical condition, they have to contain two vertices with distinct colors. New type of constraints appears with the introduction of $\mathcal{C}$-edges, which have to contain two vertices with a common color in every feasible coloring.

[1]A hypergraph consists of a vertex set and a given set of hyperedges, where hyperedge means a subset of the vertex set containing at least two vertices.

Classical hypergraphs, involving only $\mathcal{D}$-edges, always admit at least one proper color partition (in the trivial coloring every vertex has its dedicated color), and the standard problem is to determine the lower chromatic number; that is, the minimum number of colors over all proper colorings. Similarly, so-called $\mathcal{C}$-hypergraphs, containing only $\mathcal{C}$-edges, are always colorable (trivially proper coloring is obtained using just one color), but the standard question concerns the upper chromatic number; i.e., the maximum number of used colors.

The situation fundamentally changes when these two types of constraints are considered together. There exist uncolorable mixed hypergraphs with quite unrestricted structure (Tuza, Voloshin [39]), and those having only one proper color partition are also hard to recognize (Tuza, Voloshin, Zhou [40]). Moreover, surprisingly enough, mixed hypergraphs may have 'gaps' in their chromatic spectrum; that is, it can happen that a mixed hypergraph admits coloring with exactly k' and with exactly k'' colors, but there exists a natural number g between k' and k'', such that the hypergraph has no g-coloring (Jiang et al. [22]).

This observation motivated the introduction of a new term: the feasible set of a mixed hypergraph $\mathcal{H}$, denoted by $\Phi(\mathcal{H})$, is the set of integers k such that $\mathcal{H}$ admits a coloring with exactly k colors. A classical hypergraph has feasible set $\Phi(\mathcal{H}) = \{\chi(\mathcal{H}), \ldots, n\}$ without a gap, where $\chi(\mathcal{H})$ denotes the lower chromatic number and n stands for the number of vertices. Also, a $\mathcal{C}$-hypergraph has gap-free feasible set $\Phi(\mathcal{H}) = \{1, \ldots, \bar{\chi}(\mathcal{H})\}$, where $\bar{\chi}(\mathcal{H})$ denotes the upper chromatic number. As it was already mentioned, a mixed hypergraph, in general, may have gaps in the feasible set. Furthermore, it was observed by Jiang et al. [22] that every set S of positive integers not containing 1 can be obtained as the feasible set of some mixed hypergraph.

The large amount of interesting results witnesses the intensive studying of mixed hypergraph theory. They are summarized in Voloshin's research monograph [43], in the recent survey by Tuza and Voloshin [38], and on the regularly updated web site [44].

As regards applications, the complex structure of mixed hypergraphs makes it possible to handle extremal and existence problems arising in different fields.

4 New results on mixed hypergraphs

The summary of main scientific results of the Thesis is divided into two sections. Here we review contributions concerning mixed hypergraphs in two groups, based on the papers [2, 3, 7, 8].

Results, 1: Complexity of recognition and colorability

In the dissertation we study two important subclasses of mixed hypergraphs, which admit efficient coloring algorithms.

Perfection is a central concept in the classical theory of graph coloring. The class of perfect graphs contains many important subclasses and on the other hand it admits polynomial-time optimization algorithms for many problems that are NP-hard in general.

Voloshin introduced a concept in [42] that can be viewed as the dual of graph perfectness in the classical sense, and in the same paper he proposed a characterization for $\mathcal{C}$-perfect hypertrees without $\mathcal{D}$-edges.

At this point, we have to define some concepts. A colorable mixed hypergraph $\mathcal{H}$ is called $\mathcal{C}$-perfect if the largest cardinality $\alpha_C(\mathcal{H})$ of a vertex subset containing no $\mathcal{C}$-edge, equals the upper chromatic number $\bar{\chi}(\mathcal{H})$, moreover this equality holds for all induced subhypergraphs of $\mathcal{H}$. A hypertree $\mathcal{H}$ is a hypergraph for which there exists a tree graph G, such that every hyperedge of $\mathcal{H}$ induces a subtree in G. A monostar is a mixed hypergraph in which the intersection of $\mathcal{C}$-edges contains precisely one vertex. Monostars are not $\mathcal{C}$-perfect.

In the dissertation we prove that the structural characterization of $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees, conjectured since 1995, is valid indeed.

> **Contribution 1.1.** *(Corollary 3) A $\mathcal{C}$-hypertree is $\mathcal{C}$-perfect if and only if it contains no monostar as an induced subhypergraph. Moreover, $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees can be $\bar{\chi}$-colored in polynomial time.*

In the dissertation we state and prove the analogous characterization and perform $\bar{\chi}$-coloring algorithm for a wider class of $\mathcal{C}$-perfect mixed hypertrees, furthermore we observe that the characterization cannot be extended to all mixed hypertrees.

The previous theorem is supplemented with a rather unexpected complexity result.

> **Contribution 1.2.** *(Theorems 4 and 5) The following decision problems are* NP*-complete on the class of $\mathcal{C}$-hypertrees:*
>
> - *Does the hypertree $\mathcal{T}$ contain an induced monostar?*
> - *Is the hypertree $\mathcal{T}$ colorable with $\alpha_C(\mathcal{T})$ colors?*

But $\mathcal{C}$-perfect $\mathcal{C}$-hypertrees admit polynomial-time $\bar{\chi}$-coloring algorithms, moreover if this algorithm is applied for any $\mathcal{C}$-hypertree, it results in either a $\bar{\chi}$-coloring or an evidence against its $\mathcal{C}$-perfectness.

Contribution 1.3. *(Theorem 6) Over the class of $\mathcal{C}$-hypertrees there exists a polynomial-time algorithm whose output is either an induced monostar subhypergraph or a proper coloring of $\mathcal{T}$ with $\alpha_C(\mathcal{T}) = \bar{\chi}(\mathcal{T})$ colors.*

It is quite interesting to compare the above two assertions. They say that coloring is easy, recognition is hard, but nevertheless there is a concise description of the class in terms of forbidden induced substructures.

Classical hypergraphs and, similarly, $\mathcal{C}$-hypergraphs are always colorable and those having only one feasible color partition can be recognized in linear time. But the interaction of the two opposite coloring constraints allows much more complex and unrestricted structure for mixed hypergraphs having only one feasible color partition. As it was proved in [40], the recognition problem of uniquely colorable mixed hypergraphs is NP-complete even if it is restricted to colorable ones. Nevertheless, it had been expected for several years that the so-called 'UC-orderable' mixed hypergraphs are easy to recognize. 'UC-orderability' means that the vertex set of the hypergraph admits a vertex order $x_1, \ldots, x_n$ for which every hypergraph induced by a starting sequence $x_1, \ldots, x_i$ $(1 \leq i \leq n)$ is uniquely colorable. The expectation is disproved in the dissertation by the following claim.

Contribution 1.4. *(Theorem 7) Given a uniquely colorable mixed hypergraph $\mathcal{H}$ with its coloring as input, it is* NP*-complete to decide whether $\mathcal{H}$ has a UC-ordering.*

On the other hand, via a characterization theorem (Theorem 8), we prove that it can be decided in linear time whether a given color-sequence belongs to a mixed hypergraph in which the uniquely colorable vertex order is unique.[2]

Results, 2: Uniform mixed hypergraphs

Prescribing that all hyperedges have the same cardinality r, we obtain the notion of r-uniform hypergraphs. They form a widely studied subclass of hypergraphs. In the dissertation two long-standing open problems are solved concerning r-uniform mixed, and particularly bi- and $\mathcal{C}$-hypergraphs.

The possible feasible sets of mixed hypergraphs were completely characterized and the similar problem was solved also for many important subclasses. Particularly, it was shown for mixed interval hypergraphs [22], mixed hypertrees (Král', Kratochvíl, Proskurowski and Voss [25]), circular mixed hypergraphs (Král', Kratochvíl and Voss [26]) and mixed hypergraphs with maximum vertex degree two [27] that there is no gap in their feasible sets.

[2] More precisely, the order is unique apart from the transposition of the first two vertices.

However, the characterization was an open problem for two important subclasses: for r-uniform hypergraphs and also for bi-hypergraphs, where every hyperedge is a $\mathcal{C}$-edge and a $\mathcal{D}$-edge as well. In the dissertation these two questions are solved together in a constructive way. We state that there can occur gaps in these feasible sets, moreover every set S satisfying trivial necessary conditions is a feasible set of some mixed hypergraph from the considered subclasses.

> **Contribution 2.1.** *(Theorem 1) Let $r \geq 3$ be an integer, and S a non-empty finite set of positive integers. There exists an r-uniform mixed hypergraph $\mathcal{H}$ with at least one hyperedge and having feasible set $\Phi(\mathcal{H}) = S$ if and only if*
>
> *(i)* $\min(S) \geq 2$ *and S contains all integers between* $\min(S)$ *and $r-1$ (this means restriction only in the case of* $\min(S) < r-1$*), or*
>
> *(ii)* $\min(S) = 1$ *and S is of the form of $S = \{1, \ldots, \bar{\chi}\}$ for some natural number $\bar{\chi} \geq r-1$.*
>
> *Moreover, S is the feasible set of some r-uniform bi-hypergraph with $\mathcal{C} = \mathcal{D} \neq \emptyset$ if and only if it is of type (i).*

Taking into account that a bi-hypergraph with at least one hyperedge cannot be properly colored using only one color, the previous theorem implies also the following characterization: A finite set S of positive integers is the feasible set of some bi-hypergraph with at least one bi-edge if and only if $1 \notin S$.

Another open problem was to determine the minimum number of hyperedges in an r-uniform $\mathcal{C}$-hypergraph of order n, which has only trivial colorings; that is, which cannot be colored with more than $r-1$ colors (Problem 11 of [42] and Problem 2 on page 43 of [43]).

For this minimum number, in the previous literature a lower bound was given by Diao, Zhao and Zhou [15]. This bound is sharp for 3-uniform $\mathcal{C}$-hypergraphs, as it was proved by Diao, Liu, Rautenbach and Zhao [14]. But for $r > 3$, the question of tightness and even of asymptotical tightness has remained open.

It is shown in the Thesis that for every fixed $r > 3$ there exist infinitely many values of n for which the lower bound is not tight (Proposition 1). Furthermore, we prove an asymptotically tight estimate for the minimum number $f(n,r)$ of $\mathcal{C}$-edges.

Contribution 2.2. *(Theorem 2) For the minimum number $f(n,r)$ of hyperedges in an r-uniform $\mathcal{C}$-hypergraph with upper chromatic number $r-1$ the following estimates hold for all integers $n > r > 2$:*

(i) $f(n,r) \le \frac{2}{n-1}\binom{n-1}{r} + \frac{n-1}{r-1}\left(\binom{n-2}{r-2} - \binom{n-r-1}{r-2}\right)$ *for all n and r.*

(ii) $f(n,r) = (1+o(1))\,\frac{2}{r}\binom{n-2}{r-1}$ *for all $r = o(n^{1/3})$ as $n \to \infty$.*

We note that this problem can be studied in the more general context of partition-crossing set systems.[3] This observation is the starting point of a new direction of research [8, 9]. Since it is not related strongly to the other subjects of the dissertation, however, a detailed discussion is not included.

5 New models: Color-bounded and stably bounded hypergraphs

Color-bounded and stably bounded hypergraphs had not been considered before, they were introduced in our publications [1, 4, 5, 6]. These structure classes generalize the concept of mixed hypergraph; the model introduced by Drgas-Burchardt and Łazuka [16], where just the polychromatic lower bound was considered in connection with chromatic polynomials; and some earlier results [11, 29, 36], where only the monochromatic upper bound was considered from algorithmic point of view.

Introducing the notion of stably bounded (and also of color-bounded) hypergraphs, not just a common generalization of earlier coloring concepts, but as it is proved from many aspects in the dissertation, a much stronger model is obtained.

Starting with the more general concept, stably bounded hypergraph is defined as a six-tuple

$$\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}), \qquad \text{where} \qquad \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b} : \mathcal{E} \to \mathbb{N}$$

are the so-called 'color-bound functions'. (We assume that $1 \le \boldsymbol{s}(E_i) \le \boldsymbol{t}(E_i) \le |E_i|$ and $1 \le \boldsymbol{a}(E_i) \le \boldsymbol{b}(E_i) \le |E_i|$ hold for all hyperedges.) They get meaning when a vertex coloring $\varphi : X \to \mathbb{N}$ is considered. If the largest cardinality of a polychromatic and monochromatic subset of a hyperedge E_i are denoted by $\pi(E_i)$ and $\mu(E_i)$, respectively, φ is a proper coloring if it satisfies the following constraints for every hyperedge E_i:

$$\boldsymbol{s}(E_i) \le \pi(E_i) \le \boldsymbol{t}(E_i) \qquad \text{and} \qquad \boldsymbol{a}(E_i) \le \mu(E_i) \le \boldsymbol{b}(E_i).$$

[3] We say that a subset Y of the vertex set X crosses a vertex partition $\mathcal{P} = X_1 \cup \cdots \cup X_k$ if Y intersects exactly $\min(|Y|, k)$ classes of the partition $\mathcal{P}$. A set $\mathcal{H}$ is said to cross a vertex partition if so does at least one $Y \in \mathcal{H}$.

That is, polychromatic bounds $\boldsymbol{s}$ and $\boldsymbol{t}$ are responsible for the number of colors occurring on a hyperedge, whilst monochromatic bounds $\boldsymbol{a}$ and $\boldsymbol{b}$ restrict the maximum multiplicity of colors inside a hyperedge.

If a lower color-bound function ($\boldsymbol{s}$ or $\boldsymbol{a}$) has constant value 1 or an upper-bound function ($\boldsymbol{t}$ or $\boldsymbol{b}$) assigns value $|E_i|$ to each hyperedge E_i, it has no effect on the proper colorings, and it can be deleted. Taking different subsets of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$, deleting the non-restrictive conditions, the obtained structure classes are denoted with capital letters corresponding to the restrictive functions. Especially, (S,T)-hypergraph means that we have restrictive bounds only on the largest polychromatic subsets.

First we introduced and studied these (S,T)-hypergraphs, also called color-bounded hypergraphs, concentrating on the role of some special subclasses and discussing the similarities and differences compared with mixed hypergraphs.

Results, 3: Feasible sets of color-bounded hypertrees

Given a subclass of hypergraphs, the first fundamental question arising is whether or not all of its colorable members have a gap-free feasible set. It was known that every colorable mixed interval hypergraph[4] has gap-free feasible set [22] and the lower chromatic number is at most 2. In the dissertation we prove the analogous theorem for color-bounded interval hypergraphs, where s denotes the maximum value of color-bound function $\boldsymbol{s}$. Moreover, it is shown that these statements are valid also for Rooted Directed Path hypertrees (where every edge of the host tree is directed away from the root and every hyperedge corresponds to a directed path in it).

> **Contribution 3.1.** *(Theorems 14 and 15) Every colorable color-bounded interval hypergraph and Rooted Directed Path hypergraph has a gap-free chromatic spectrum, and its lower chromatic number is equal to s.*

Significant differences appear when we consider hypertrees in general. Whilst mixed hypertrees can have only those feasible sets as mixed interval hypergraphs, in the color-bounded model hypertrees play central role regarding colorability properties. Although they have quite restricted structure, it turned out that they represent nearly all color-bounded hypergraphs concerning feasible sets. The characterization of possible feasible sets contains only one additional restriction to the general case: every two-colorable color-bounded hypertree necessarily has a gap-free feasible set.

[4] An interval hypergraph is a hypertree whose host tree is a path.

Contribution 3.2. *(Theorems 16 and 17) Let S be a finite set of positive integers. There exists a color-bounded hypertree $\mathcal{T}$ with feasible set $\Phi(\mathcal{T}) = S$ if and only if*

(i) $\min(S) = 1$ *or* $\min(S) = 2$*, and S contains all integers between* $\min(S)$ *and* $\max(S)$*, or*

(ii) $\min(S) \geq 3$.

The characterization is valid also for the class of r-uniform color-bounded hypertrees, for each integer $r \geq 4$. Moreover, it can be supplemented with a stronger proposition stating that for any possible feasible set S satisfying $\min(S) \geq 3$, for every integer $k \in S$, the number of proper k-color partitions can be arbitrarily prescribed (Corollary 14).

The dissimilarity between mixed and color-bounded hypertrees concerns not only feasible sets, but also time complexity of the colorability problem. The colorability of mixed hypertrees can be decided in linear time [39], but the corresponding problem was proven NP-complete not only on color-bounded hypertrees in general, but already on the 3-uniform ones, as it will be stated in Contribution 4.2.

On the other hand, we determined some subclasses of color-bounded hypergraphs where the colorability problem can be solved in polynomial time, e.g., for 3-uniform interval hypergraphs, for hypergraphs without overlapping edges and for hypergraphs whose intersection graph is a tree.

Circular color-bounded hypergraphs are also investigated in the dissertation and it is proved that their possible feasible sets are fairly restricted (Theorem 19).

Results, 4: Colorability of stably bounded hypergraphs

Returning to stably bounded hypergraphs, the main results concern the comparison of different models, which can be obtained by choosing some of the four color-bound functions $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{a}$, $\boldsymbol{b}$ as restrictive bounds.

The upper-bound functions $\boldsymbol{t}$ and $\boldsymbol{b}$ always can be reduced to $\boldsymbol{a}$ and $\boldsymbol{s}$, respectively, if structural conditions (e.g., hypertree, uniformity) are not imposed. But the lower-bound functions $\boldsymbol{s}$ and $\boldsymbol{a}$ generally cannot be expressed with $\boldsymbol{t}$ and $\boldsymbol{b}$. by the other ones. In this sense, the model (S,A) is the only universal pair.

A (T,A)- and also (S,B)-hypergraphs are always colorable and have gap-free feasible sets. All of the remaining pairs may admit uncolorability and gaps in the feasible set. But there are differences between these models regarding the minimum order of a hypergraph that can have a gap of size k.[5]

Contribution 4.1. *(Theorems 10, 20 and 21)*

- *If a (T, A, B)-, (T, B)- or (A, B)-hypergraph has a gap of size $k \geq 1$ in its feasible set, then it has at least $2k + 4$ vertices.*
- *If an (S, A)-, (S, T)-, or stably bounded hypergraph has a gap of size $k \geq 1$ in its feasible set, then it has at least $k + 5$ vertices.*

Moreover, all the above bounds are sharp.

The colorability problem was proved to be NP-complete already on 3-uniform (S, T)-hypertrees. This result can be extended for all of our models which admit uncolorability.

Contribution 4.2. *(Theorems 18 and 23) The decision problem of colorability is* NP*-complete on each of the following classes of hypergraphs:*

- *3-uniform (S, T)-hypertrees,*
- *3-uniform (S, A)-hypertrees,*
- *3-uniform (T, B)-hypertrees,*
- *3-uniform (A, B)-hypertrees.*

Although it is hard to test whether a mixed hypergraph is uniquely colorable, this is not the case if $\bar{\chi}$ is very near to the number n of vertices. For the latter case Niculitsa and Voss described characterizations of uniquely $(n-1)$- and $(n-2)$-colorable mixed hypergraphs [33] which result in polynomial time algorithms. In contrast to this, we prove in the dissertation that the recognition problem of uniquely $(n-1)$-colorable (S,A)- and (S,T)-hypergraphs is hard. But we can give (quite technical) characterizations for uniquely $(n-1)$-colorable (T,B)- and (A,B)-hypergraphs, which yield polynomial-time recognition algorithms.

[5] If the feasible set contains integers k' and k'', for which $k' + 1 < k''$ holds, but no integer between them, then the size of the gap is defined as $k = k'' - k' - 1$.

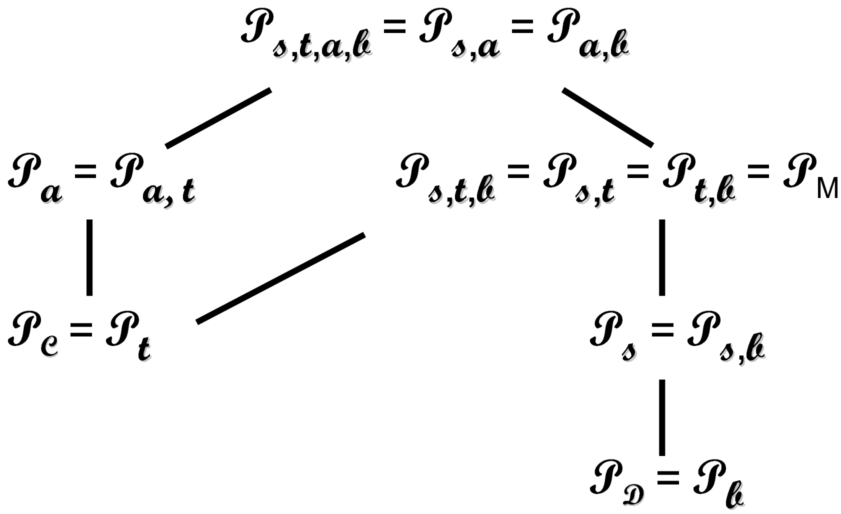

Figure 1: Hasse-diagram of possible chromatic polynomials belonging to different structure classes. The letters $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ stand for the color-bound functions, while $\mathcal{C}, \mathcal{D}$ and M denote $\mathcal{C}$-, $\mathcal{D}$- and mixed hypergraphs, respectively.

Contribution 4.3. *(Theorems 12, 24 and 27)*

- *The decision problem of unique* $(n-1)$*-colorability is* co-NP*-complete for (S,A)-, (S,T)-, and stably bounded hypergraphs.*
- *The decision problem of unique* $(n-1)$*-colorability can be solved in polynomial time for (T,A,B)-, (T,B)- and (A,B)-hypergraphs.*

Results, 5: Chromatic polynomials

In the last Thesis, I would like to emphasize a theorem from the dissertation which concerns mixed, color-bounded and stably bounded hypergraphs as well and also establishes connection with another part of mathematics.

The chromatic polynomial of a graph or a hypergraph $\mathcal{H}$ is defined as the polynomial $P(\mathcal{H}, \lambda)$ whose value for every integer $\lambda = k$ is the number of feasible colorings of $\mathcal{H}$ with at most k colors. (Here the permutations of colors are counted to be distinct.) As usual, for $n \geq k > 0$, the Stirling number of second kind is denoted by $S(n, k)$.[6]

[6]Stirling number of second kind $S(n, k)$ denotes the number of partitions of n elements into precisely k nonempty classes.

Contribution 5. *(Theorem 9) Let* $P(\lambda) = \sum_{k=0}^{\ell} a_k \lambda^k \not\equiv 0$ *be a polynomial such that* $P(1) = 0$*, i.e.* $\sum_{k=0}^{\ell} a_k = 0$*. The following properties are equivalent.*

1. $P(\lambda)$ *is the chromatic polynomial of a stably bounded hypergraph.*
2. $P(\lambda)$ *is the chromatic polynomial of a color-bounded hypergraph.*
3. $P(\lambda)$ *is the chromatic polynomial of a mixed hypergraph.*
4. $P(\lambda)$ *satisfies all of the following conditions.*
   - *(i) All coefficients* a_k *of* $P(\lambda)$ *are integers.*
   - *(ii) The leading coefficient* a_ℓ *is positive.*
   - *(iii) The constant term* a_0 *is zero.*
   - *(iv) For every positive integer* $j \leq \ell$*, the inequality*

$$\sum_{k=j}^{\ell} a_k \cdot S(k, j) \geq 0$$

     is valid.

Due to the condition $P(1) = 0$, this theorem concerns only the non-one-colorable hypergraphs of listed types. In the Thesis we compare the possible chromatic polynomials without this assumption, and obtain that (S,A)-, (A,B)- and stably bounded hypergraphs have the same set of possible polynomials, containing the more restricted set belonging to (S,T)-, (T,B)- and mixed hypergraphs (Theorem 22, Propositions 31 and 32). These connections are demonstrated on the Hasse-diagram of Figure 1.

Acknowledgement

First of all, I would like to give thanks to Professor Zsolt Tuza for the supervision. His exceptional competence and excellent guidance gave me powerful help to advance in the research.

I kindly acknowledge to University of Pannonia and especially to Professor Ferenc Friedler for the opportunity and for the supporting environment provided for my research.

Thanks are due to my colleagues at the Department of Computer Science and to my colleagues in Nagykanizsa for their help. Furthermore, I would like to thank the participants of the seminar at the Research Group for Fault-Tolerant Systems (BUTE) for discussions on applications of hypergraphs.

I thank to my family for the encouragement, and finally, I acknowledge the financial support of the Hungarian Scientific Research Fund (OTKA T-049613).

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I. Publications from the results of the Thesis

[1] Cs. Bujtás and Zs. Tuza: Mixed colorings of hypergraphs. *Electronic Notes in Discrete Mathematics* 24 (2006), 273–275.

[2] Cs. Bujtás and Zs. Tuza: Orderings of uniquely colorable hypergraphs. *Discrete Applied Mathematics* 155 (2007), 1395–1407. (Impact Factor 0.577)

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