Response to the questions from the referee report and review of opponent Prof. József Békési

Dear Prof. József Békési,

Thank you for your careful evaluation of my doctoral dissertation and for your insightful questions. Your questions address two central aspects of my research:

1. The computational limits of exact MILP formulations for the Board Packing Problem (BoPP) and the potential of alternative paradigms such as Constraint Programming, and

2. The modelling and algorithmic approaches that can best determine whether squares of consecutive sizes can be packed into a larger square.

Below, I provide my detailed answers, which are based on the theoretical framework, empirical results, and published papers forming the foundation of my dissertation.

Answer to Question 1

Question: *„The computational experiments in Chapter 2 shows that CPLEX gives „Out of memory" error in many cases. What is your opinion on the possibilities of using other types of methods, such as Constraint Programming, for the problems examined?"*

Answer. The exact methods applied in my dissertation are primarily based on **Mixed–Integer Linear Programming (MILP)**. While MILP formulations, implemented in CPLEX, prove extremely effective for small and medium-sized instances of the Board Packing Problem (BoPP), my experiments in Chapter 2 demonstrate that they do not scale to large instances. This is a natural consequence of the NP-hardness of BoPP: as the number of potential placements grows combinatorially, the branch-and-bound tree and the number of binary variables quickly exceed available memory.

A promising complementary paradigm is **Constraint Programming (CP)**, which models logical and geometric relations directly and exploits constraint propagation to prune infeasible configurations early. In the related literature, CP has already shown strong performance on two-dimensional packing tasks.

For instance, Clautiaux et al. (2007, 2008) proposed CP models for the orthogonal rectangle-packing problem using cumulative scheduling and energetic reasoning, achieving fast feasibility detection and strong pruning. Similarly, Soh (2008) developed a concise CP model for packing consecutive squares, using a single no-overlap disjunctive constraint that is structurally simpler than a MILP formulation.

However, BoPP is not a pure non-overlap packing problem; it is a profit-maximization problem where overlap is permitted but the revenue of each cell is counted only once. In this setting, CP alone cannot fully replace MILP, but it can complement it efficiently within a hybrid architecture:

1. **MILP (CPLEX)** provides tight lower and upper bounds and exact proofs for small/medium boards.

2. **CP or CP-based Large Neighborhood Search (LNS)** can refine or repair placements and explore large feasible neighborhoods through constraint propagation.

3. **The Evolutionary Algorithm (EA)** developed in my dissertation effectively handles large-scale profit optimization once exact solvers hit memory limits.

Within such a MILP + CP + EA hybrid pipeline, CP can be used

- to eliminate infeasible regions and overlapping conflicts through propagation,
- as a pre-processing filter or consistency checker for MILP,
- or as a feasibility improver inside a metaheuristic framework.

Furthermore, CP becomes particularly efficient in BoPP variants without overlap, such as the square-packing and covering subproblems discussed in Chapter 3. In those cases, CP models using the global `diffn`-constraint ensures that all rectangles occupy distinct, non-overlapping positions in space), or no-overlap constraints can outperform MILP because the geometric relationships are directly encoded as disjunctions instead of binary variables.

In summary, while MILP remains ideal for exact profit optimization on small instances, Constraint Programming, possibly combined with LNS or other metaheuristics, offers the best path forward for handling large, combinatorial BoPP instances.

This hybrid perspective is also consistent with recent follow-up work citing our BoPP (EJOR 2023) paper, where Variable Neighborhood Search (VNS) metaheuristics achieved results comparable to our EA, confirming that no single exact paradigm can scale unboundedly.

Answer to Question 2

Question: *„The third chapter of the dissertation discusses the square packing problem. What do you think, what kind of computer models or algorithms could be the best to decide whether, for example, small squares of consecutive sizes can be packed into a larger square?"*

Answer. The *square packing problem* studied in Chapter 3 asks whether squares of sizes $1 \times 1, 2 \times 2, \ldots, n \times n$ can be placed, without rotation or overlap, inside a square of side K. This decision version: *„Does a feasible packing exist for a given K?"* is NP-hard and serves as a benchmark problem in discrete geometry and combinatorial optimization (OEIS A005842).

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In my dissertation, the problem was approached mainly through **Mixed–Integer Linear Programming (MILP)** models solved by CPLEX, supported by theoretical and heuristic analyses rather than by Constraint Programming. Nevertheless, **Constraint Programming (CP)** is known to be a strong alternative for feasibility-oriented geometric problems of this type. CP models using the global `diffn` (no-overlap) constraint ensure that all rectangles occupy distinct, non-overlapping positions in space. Because the pairwise disjunctions are handled natively by propagation, CP can outperform MILP on feasibility pruning. While CP was not implemented in my work, it represents a promising direction for future research, especially for moderate-size instances where pure MILP becomes computationally demanding.

1. **Mixed–Integer Linear Programming (MILP).** MILP formulations were employed to verify or exclude specific values of K. In particular, CPLEX confirmed infeasibility for $n = 18$ at $K = 46$ (and feasibility at 47) and verified bounds up to about $n \approx 25$. MILP relaxations also provided informative numerical bounds complementing the theoretical lower limits.

2. **Heuristic and Asymptotic Methods.** For larger n, theoretical and constructive algorithms were developed. The paper *Guillotine cutting is asymptotically optimal for packing consecutive squares* (*Optimization Letters*, 2022) proves that all squares $[1, \ldots, n]$ can be packed into a square of side $N + cn$ for some constant $c < 1$, and even tighter $N + (\frac{1}{2} + \varepsilon)n + O(\sqrt{n})$ when guillotine cuts are relaxed. These asymptotic results provide both upper bounds and constructive layouts that exact solvers can refine.

3. **Future Directions.** Although CP was not part of the implemented methods in this dissertation, it could serve as a complementary approach in future studies. By integrating MILP for bounding, CP for feasibility propagation, and heuristic search for large instances, one may obtain a scalable hybrid framework for solving or approximating difficult square–packing instances.

To decide whether a packing exists for given n and K, the recommended workflow is therefore:

1. Apply analytical lower bounds ($LB_0 = \lceil \sqrt{\sum k^2} \rceil$, $LB_1 = n + (n - 1)$, $LB_2 = 3n - 9$) to prune impossible K;

2. Use MILP for exact verification on small and medium instances;

3. Employ heuristic or asymptotic constructions to explore larger instances;

4. Consider CP models as a natural extension for feasibility-oriented future research.

Beyond these three methodological directions: **Mixed–Integer Linear Programming**, **Constraint Programming**, and **heuristic or asymptotic constructions**, there are no fundamentally different paradigms currently capable of addressing the square–packing problem effectively. Alternative techniques, such as

SAT/SMT formulations, **metaheuristics** (e.g., Genetic Algorithms, Tabu Search, or Variable Neighborhood Search), or **machine–learning–assisted search**, can be regarded as extensions or hybrids of the above categories. Therefore, the framework presented in my dissertation already spans the complete range of practical and theoretically justified methods for tackling this class of geometric packing problems.

Key References

- Abraham, G., Dósa, G., Hvattum, L. M., **Olaj, T. A.**, Tuza, Zs. (2023). *The Board Packing Problem. European Journal of Operational Research*, 308(3), 1056–1073.
- Balogh, J., Dósa, G., Hvattum, L. M., **Olaj, T. A.**, Tuza, Zs. (2022). *Guillotine cutting is asymptotically optimal for packing consecutive squares. Optimization Letters*, 16, 2775–2785.
- Balogh, J., Dósa, G., Hvattum, L. M., **Olaj, T. A.**, Szalkai, I., Tuza, Zs. (2025). *Covering a square with consecutive squares. Annals of Operations Research* (Springer).
- Clautiaux, F., Carlier, J., Moukrim, A. (2007). *A new exact method for the two-dimensional orthogonal packing problem. European Journal of Operational Research*, 183(3), 1196–1211. https://doi.org/10.1016/j.ejor.2005.12.048
- Clautiaux, F., Jouglet, A., Carlier, J., Moukrim, A. (2008). *A new constraint programming approach for the orthogonal packing problem. Computers & Operations Research*, 35(3), 944–959. https://doi.org/10.1016/j.cor.2006.05.012
- Côté, J.-F., Iori, M. (2016). *The meet-in-the-middle principle for cutting and packing problems (CIRRELT-2016-28). CIRRELT.*
- Côté, J.-F., Iori, M. (2018). *The meet-in-the-middle principle for cutting and packing problems. INFORMS Journal on Computing*, 30(4), 646–661. https://doi.org/10.1287/ijoc.2018.0806
- Kovács, A. (2004). *A korlátozás programozás alapjai.* Budapest University of Technology and Economics, Mesterséges Intelligencia Labor.
- Soh, T. (2008). *Packing consecutive squares into a square.* Kobe University.
- Soh, T., Inoue, K., Tamura, N., Banbara, M., Nabeshima, H. (2008).*A SAT-based method for solving the two-dimensional strip packing problem. In Proceedings of the 15th International RCRA Workshop on Experimental Evaluation of Algorithms for Solving Problems with Combinatorial Explosion*, pp. 1–15. Udine, Italy.

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