Response to the Questions from the Referee Report and Review of Opponent Dr. Tibor Dulai

Dear Dr. Tibor Dulai,

Thank you for your careful evaluation of my doctoral dissertation and for your insightful questions.

Answer to Questions

Questions: *"Can your solutions to the BoPPs and Square Packing problems be modified to solve 3D problems? If so, how?"*

Yes. Both the **Board Packing Problem (BoPP)** and the **Square Packing Problem (SPP)** can be naturally extended from two to three dimensions. In the literature, a well-known and well-studied model is the *Three-Dimensional Bin Packing Problem.* Martello, Pisinger, and Vigo [1] characterize this problem as "strongly NP-hard and extremely difficult to solve in practice" (Abstract). When a single fixed container is considered, the problem is referred to by the authors as the *Container Loading Problem.* This model is first introduced in a heuristic form by George and Robinson [2] and later evaluated comparatively by Bischoff and Marriott [3]. Although our 3D BoPP is not identical to the classical 3D Bin Packing Problem, it has a similar combinatorial structure and therefore belongs to the same NP-hard class.

Regarding the 3D version of BoPP

Note that there are two main versions of the BoPP model, where overlap is allowed, or it is forbidden. Here I will focus only on the case where overlap is not allowed, but the other case can be treated similarly.

Thus, the 3D version considered here is the following: Given boxes with three dimensions (w_r, d_r, h_r). Given also a 3D board of size (M, N, H)3D. Each cell has a given gain value, and each box has an associated cost c_r.

Then the goal is to buy several boxes and place them into the 3D board so that

- there is no overlap
- the total profit is maximized. The profit is calculated so that the total gain value that is got for the covered 3D cells, minus the total cost of purchasing the chosen boxes. Because the number of feasible placements grows cubically with instance size, the 3D version is computationally very demanding.

Solution with a MILP or CP model.

The decision variables $x_{i,j,k,r}$ are binary and equal to 1 if box r is placed with its lower-left-front corner at (i, j, k) and 0 otherwise. In the objective function the total gain is calculated, similarly to the $2D$ version. Boundary constraints ensure that no box extends beyond the container:

$$x_r + w_r \leq M, \qquad y_r + d_r \leq N, \qquad z_r + h_r \leq H.$$

For any pair of boxes (r, s), at least one of the following non-overlap conditions must be true:

$$(x_r{+}w_r \leq x_s)\ ,\ (x_s{+}w_s \leq x_r)\ ,\ (y_r{+}d_r \leq y_s)\ ,\ (y_s{+}d_s \leq y_r)\ ,\ (z_r{+}h_r \leq z_s)\ ,\ (z_s{+}h_s \leq z_r)$$

which can be guaranteed by binary variables and constraints in the MILP model or a CP condition in the CP model. Since the number of potential placements grows very fast, it can be conjectured that for medium-sized model the solution with a MILP solver (like CPLEX) does not seem a good idea.

Evolutionary algorithm.

The evolutionary evaluation algorithm originally developed for the 2D BoPP can be extended naturally to 3D. Each candidate solution is encoded by the coordinates (x_r, y_r, z_r) of every box and a binary selection flag. Feasibility is verified through the 3D non-overlap constraints, while local search operations move or swap boxes along the three coordinate axes. This process efficiently generates high-quality feasible solutions that can be used as initial incumbents or for comparison with the MILP solver, integrating exact and heuristic perspectives within a unified framework.

The main steps: initialization, construction, local improvement, evaluation, and selection, are similar to those in the 2D algorithm, except that all spatial operations are extended from (x, y) to (x, y, z). An overview of the 2D algorithm, whose components are easily adaptable to the 3D case, is shown in Figure 1.

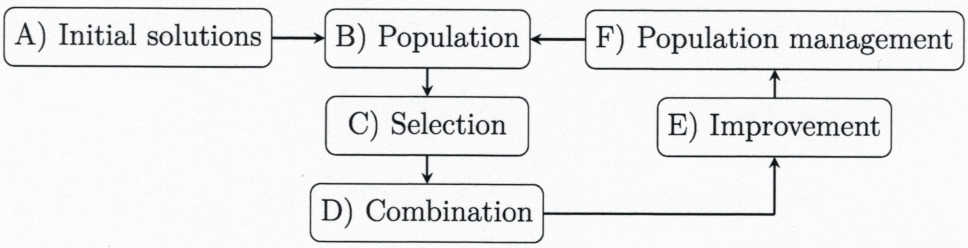

Figure 1: Overview of the evolutionary search framework developed for the BoPP. Each component can be adapted to the three-dimensional case without major modifications.

In conclusion, the three-dimensional BoPP is a direct extension of the $2D$ model. The model captures the same decision logic, extends the non-overlap constraints to three dimensions, and remains strongly NP-hard.

3D Cube Packing Problem (CPP)

The two-dimensional Square Packing Problem (SPP) can be naturally extended to three dimensions as the *Cube Packing Problem (CPP)*, where cubes of sides

$1, 2, \ldots, n$ are packed without overlap and rotation into the smallest possible enclosing cube of side K. Note that the 2D model is listed in [4], in the On-Line Encyclopedia of Integer Sequences, as A005842, but it seems the 3D model is never investigated yet.

MILP model

The MILP formulation of the CPP can be directly derived from the 3D Board Packing Problem (BoPP). Let (x_r, y_r, z_r) denote the lower–left–front corner of cube r with side length w_r. Containment within the cube of side K is given by:

$$x_r + w_r \leq K, \qquad y_r + w_r \leq K, \qquad z_r + w_r \leq K.$$

For any two cubes (r, s), let binary variables $u^x_{r,s}, u^y_{r,s}, u^z_{r,s} \in \{0, 1\}$ indicate whether cube r lies to the left, in front, or below cube s. Then we have

$$\begin{aligned} x_r + w_r &\leq x_s + M(1 - u^x_{r,s}) \\ y_r + w_r &\leq y_s + M(1 - u^y_{r,s}) \\ z_r + w_r &\leq z_s + M(1 - u^z_{r,s}) \end{aligned}$$

where M is a large number. Then at least one relation holds among the following six relations (for any two cubes r and s):

Cube r is to the left from cube s, or cube s is to the left from cube r; or cube r is to in front of cube s, or cube s is to in front of cube r; or cube r is above cube s, or cube s is above cube r.

Thus, we have: $$u^x_{r,s} + u^x_{s,r} + u^y_{r,s} + u^y_{s,r} + u^z_{r,s} + u^z_{s,r} \geq 1, \quad \text{for any } r \neq s$$

This formulation ensures that for each pair of cubes, at least one separating plane exists along one of the coordinate axes. The objective is to minimize K, i.e., to find the smallest enclosing cube satisfying these constraints.

Lower bounds

The analytical lower bounds proposed for SPP can be extended directly to the CPP:

$$LB_0 = \left\lceil \sqrt[3]{\sum_{i=1}^{n} i^3} \right\rceil = \left\lceil \sqrt[3]{\tfrac{n^2(n+1)^2}{4}} \right\rceil, \qquad LB_1 = 2n - 1, \qquad LB_2 = 3n - 21.$$

Here LB_0 is the *volume bound* derived from total cube volume, LB_1 separates the two largest cubes by one plane, and LB_2 follows from the geometric fact that among any nine axis-parallel, non-overlapping cubes, at least three must be aligned (collinear) along one of the coordinate axes (see Figure 2).

The third lower bound is the most interesting here, and it can be proven in the following way.

Claim 1. *Let us take any 9 cubes, without rotation and overlap. Then, we find at least 3 cubes among the 9 cubes that are collinear.*

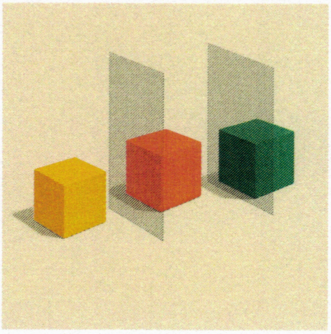

Figure 2: Collinear cubes: at least three of nine non-overlapping cubes are aligned (collinear) along one of the coordinate axes.

Remark 1. *Here "collinear" means that they are separated by some planes that are parallel with the x and y; or x and z; or with y and z axes. Note that if the claim holds, then the lower bound comes as the sum of sizes of the smallest three cubes among the nine biggest cubes.*

Proof. Let us create an oriented complete graph with 9 vertices, each vertex corresponding to some cube. For any two cubes i and j, we orient the edge from i to j and color the edge with color red, if i is to the left of j, with color blue, if i is beyond j, with color black, if i is above j. We need to show that there exists a directed path of two edges with the same color in the graph. Now let us omit the orientation, we have a complete graph colored by three colors. Let us consider the black edges. If we have a cycle with odd number of edges, we are done, as this cycle will contain the required path. Otherwise the subgraph of the black edges is bipartite, $G(A, B, E)$. Since there are 9 vertices, one of the vertex sets A and B has at least 5 vertices, say this is set A. There is no black edge between any vertices of A. So this subgraph is colored by the other two colors, and there is already shown for such a graph in the thesis that it must contain the desired path. ∎

Note that the claim is not true for 8 cubes. In the above way we do have three lower bounds, all of these lower bounds are the 3D versions of our previous lower bounds of the 2D case.

Calculations

Solving the model for $1 \leq n \leq 20$, we obtain the following results (bold values indicate the largest bound). One can see that LB_1 dominates LB_0 and LB_2 for $2 \leq n \leq 20$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
LB_0	**1**	**3**	4	5	7	8	10	11	13	15	17	19	21	23	25	27	29	31	34	36
LB_1	–	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**21**	**23**	**25**	**27**	**29**	**31**	**33**	**35**	**37**	**39**
LB_2	–	–	–	–	–	–	–	–	–	6	9	12	15	18	21	24	27	30	33	**39**
K	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39

For $21 \leq n \leq 40$, LB_2 becomes stronger than both LB_1 and LB_0. The following table lists only LB_2 for this range of n.

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
LB_2	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87	90	93	96	99
K	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87	90	93	96	99

From the two tables above, the tight results for all $1 \leq n \leq 40$ are already determined.

The analytical lower bounds behave consistently also for larger n: for $n = 50$, the strongest bound is $LB_2 = 3 \cdot 50 - 21 = 129$, while for $n = 100$ the dominant bound becomes

$$LB_0 = \left\lceil \sqrt[3]{\frac{100^2 \cdot 101^2}{4}} \right\rceil = 295.$$

The corresponding minimal cube side K for these higher instances has not yet been determined; ongoing computational experiments (using GAMS/CPLEX) aim to obtain tight values for large n.

References

[1] Martello, S., Pisinger, D., & Vigo, D. (2000, March–April). *The Three-Dimensional Bin Packing Problem. Operations Research*, 48(2), 256–267.

[2] George, J. A., & Robinson, D. F. (1980). *A heuristic for packing boxes into a container. Computers & Operations Research*, 7(3), 147–156.

[3] Bischoff, E. E., & Marriott, M. D. (1990). *A comparative evaluation of heuristics for container loading. European Journal of Operational Research*, 44(2), 267–276.

[4] The On-Line Encyclopedia of Integer Sequences, A005842. `http://oeis.org`

Place: Fredrikstad, Norway
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Tomas Attila Olaj
(Signature)